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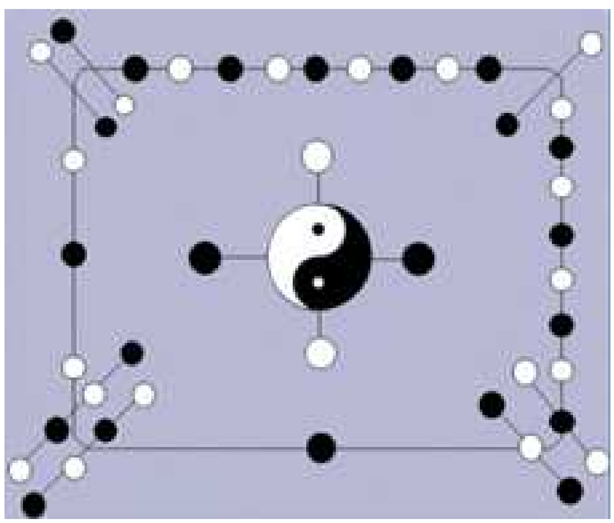
Edited By Linfan MAO



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There are many wonderful things in nature, but the most wonderful of all is man.

By Sophocles, an ancient Greek dramatist.

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Tangent Space and Derivative Mapping on Time Scale

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Abstract: In this paper, considering the time scale concept, we introduce the tangent vector and some properties according to directional derivative, the delta differentiable vector fields on regular curve parameterized by time scales and the Jacobian matrix of -completely delta differentiable two variables function.

Key Words: Time Scale, regular curve, derivative mapping.

AMS(2000): 53B30, 51B20.

§1. Introduction

The calculus of time scales was introduced by Aulbach and Hilger [1,2].This theory has proved to be useful in the mathematical modeling of several important dynamic processes [3,4,5]. We know that the directional derivative concept is based on for some geometric and physical investigations. It is used at the motion according to direction of particle at the physics [6]. Then, Bohner and Guseinov has been published a paper about the partial differentiation on time scale [7]. Here, authors introduced partial delta and nabla derivative and the chain rule for multivariable functions on time scale and also the concept of the directional derivative. Then, the directional derivative according to the vector field has defined [8]. The general idea in this paper is to investigate some properties of directional derivative. Then, using the directional derivative, we define tangent vector space and delta derivative on vector fields. Finally, we write Jacobian matrix and the -derivative mapping of the -completely delta differentiable two variables functions. So our intention is to use several new concepts, which are defined in differential geometry [7].

§2. Partial differentiation on time scale

Let $n \in N$ be fixed. Further, for each $i \in \{1, 2, \cdots, n\}$ let T_i denote a time scale, that is, T_i is a nonempty closed subset of the real number $\mathbf{R}$. Let us set

$$\Lambda^n = T_1 x T_2 x \cdots x T_n = \{t = (t_1, t_2, \cdots, t_n) \text{ for } t_i \in T_i,\ 1 \le i \le n\}$$

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We call Λ^n an n-dimensional time scale. The set Λ^n is a complete metric space with the metric d defined by

$$d(t,s)=\sqrt{\sum_{i=1}^{n}|t_i-s_i|^2}, \quad \text{for } t,s\in\Lambda^n.$$

Let σ_i and ρ_i denote , respectively, the forward and backward jump operators in T_i. Remember that for $u\in T_i$ the forward jump operator $\sigma_i:T_i\to T_i$ is defined by

$$\sigma_i(\mu)=inf\{\nu\in T_i:\ \nu>\mu\}$$

and the backward jump operator $\rho_i:T_i\to T_i$ by

$$\rho_i(\mu)=inf\{\nu\in T_i:\ \nu<\mu\}.$$

In this definition we put $\sigma_i(\max T_i)=\max T_i$ if T_i has a finite maximum, and $\rho_i(\min T_i)=\min T_i$ if T_i has a finite minimum. If $\sigma_i(\mu)>\mu$, then we say that μ is right-scattered (in T_i), while any μ with is left-scattered (in T_i). Also, if $u<\max T_i$ and $\sigma_i(\mu)=\mu$, then μ is called right-dense (in T_i), and if $\mu>\min T_i$ and $\rho_i(\mu)=\mu$, then μ is called left- dense (in T_i). If T_i has a left-scattered minimum m, then we define $T_i^k=T_i-\{m\}$, otherwise $T_i^k=T_i$. If T_i has a right-scattered maximum M, then we define $(T_i)_k=T_i-\{M\}$, otherwise $(T_i)_k=T_i$.

Let $f:\Lambda^n\to\mathbf{R}$ be a function. The partial delta derivative of f with respect to $t_i\in T_i^k$ is defined as the limit

$$\lim_{\substack{s_i\to t_i\\ s_i\neq\sigma_i(t_i)}}\frac{f(t_1,\cdots,t_{i-1},\sigma_i(t_i),t_{i+1},\cdots,t_n)-f(t_1,\cdots,t_{i-1},s_i,t_{i+1},\cdots,t_n)}{\sigma_i(t_i)-s_i}=\frac{\partial f(t)}{\Delta_i t_i}.$$

Definition 2.1 *We say a function $f:\Lambda^n\to\mathbf{R}$ is completely delta differentiable at a point $t^0=(t_1^0,t_2^0,\cdots,t_n^0)\in T_1^kxT_2^kx\cdots xT_n^k$ if there exist numbers $A_1,A_2,\cdots,A_n$ independent of $t=(t_1,t_2,\cdots,t_n)\in\Lambda^n$ (but in general, dependent on t^0) such that for all $t\in U_\delta(t^0)$,*

$$f(t_1^0,t_2^0,\cdots,t_n^0)-f(t_1,t_2,\cdots,t_n)=\sum_{i=1}^{n}A_i(t_i^0-t_i)+\sum_{i=1}^{n}\alpha_i(t_i^0-t_i)$$

and, for each $j\in\{1,2,\cdots,n\}$ and all $t\in U_\delta(t^0)$,

$$\begin{aligned}&f(t_1^0,\cdots,t_{j-1}^0,\sigma_j(t_j^0),t_{j+1}^0,\cdots,t_n^0)-f(t_1,\cdots,t_{j-1},t_j,t_{j+1},\cdots,t_n)\\&=A_j(\sigma_j(t_j^0)-t_j)+\sum_{\substack{i=1\\i\neq j}}^{n}A_i(t_i^0-t_i)+\beta_{jj}(\sigma_j(t_j^0)-t_j)+\sum_{\substack{i=1\\i\neq j}}^{n}\beta_{ij}(t_i^0-t_i),\end{aligned}\qquad(2-1)$$

where δ is a sufficiently small positive number, $U_\delta(t^0)$ is the δ-neighborhood of t^0 in Λ^n, $\alpha_i=\alpha_i(t^0,t)$ and $\beta_{ij}=\beta_{ij}(t^0,t)$ are defined on $U_\delta(t^0)$ such that they are equal to zero at $t=t^0$ and such that

$$\lim_{t \to t^0} \alpha_i(t^0, t) = 0 \ and \ \lim_{t \to t^0} \beta_{ij}(t^0, t) = 0 \ for \ all \ i, j \in \{1, 2, \cdots, n\}.$$

Definition 2.2 *We say that a function $f : T_1 x T \to \mathbf{R}$ is σ_1-completely delta differentiable at a point $(t^0, s^0) \in T_1 x T$ if it is completely delta differentiable at that point in the sense of conditions (2.5)-(2.7)(see in [7]) and moreover, along with the numbers A_1 and A_2 presented in (2.5)-(2.7) (see in [7]) there exists also a number B independent of $(t, s) \in T_1 x T$ (but, generally dependent on (t^0, s^0)) such that*

$$\begin{aligned} & f(\sigma_1(t^0), \sigma_2(s^0)) - f(t, s) \\ & = A_1(\sigma_1(t^0) - t) + B(\sigma_2(s^0) - s) + \gamma_1(\sigma_1(t^0) - t) + \gamma_2(\sigma_2(s^0) - s) \end{aligned} \qquad (2-2)$$

for all $(t, s) \in V^{\sigma_1}(t^0, s^0)$, a neighborhood of the point (t^0, s^0) containing the point $(\sigma_1(t^0), s^0)$, and the functions $\gamma_1 = \gamma_1(t^0, s^0, t, s)$ and $\gamma_2 = \gamma_2(t^0, s^0, t, s)$ are equal to zero for $(t, s) = (t^0, s^0)$ and

$$\lim_{(t,s) \to (t^0, s^0)} \gamma_1 = \gamma_1(t^0, s^0, t, s) \ and \ \lim_{s \to s^0} \gamma_2(t^0, s^0, s) = 0. \qquad (2-3)$$

Note that in $(2-1)$ the function γ_2 depends only on the variable s. Setting $s = \sigma_1(s^0)$ in $(2-2)$ yields

$$B = \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s}.$$

Furthermore, let two functions

$$\varphi : T \to \mathbf{R} \quad \text{and} \quad \psi : T \to \mathbf{R}$$

be given and let us set

$$\varphi(T) = T_1 \quad \text{and} \quad \psi(T) = T_2.$$

We will assume that T_1 and T_2 are time scales. Denote by σ_1, Δ_1 and σ_2, Δ_2 the forward jump operators and delta operators for T_1 and T_2, respectively. Take a point $\xi^0 \in T^k$ and put $t^0 = \varphi(\xi^0)$ and $s^0 = \psi(\xi^0)$.

We will also assume that

$$\varphi(\sigma(\xi^0)) = \sigma_1(\varphi(\xi^0)) \text{ and } \psi(\sigma(\xi^0)) = \sigma_2(\psi(\xi^0)).$$

Under the above assumptions let a function $f : T_1 x T_2 \to \mathbf{R}$ be given.

Theorem 2.1 *Let the function f be σ_1-completely delta differentiable at the point (t^0, s^0). If the functions φ and ψ have delta derivative at the point ξ^0, then the composite function*

$$F(\xi) = f(\varphi(\xi), \psi(\xi)) \ for \ \xi \in T$$

has a delta derivative at that point which is expressed by the formula

$$F^{\Delta}(\xi^0) = \frac{\partial f(t^0, s^0)}{\Delta_1 t}\varphi^{\Delta}(\xi^0) + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s}\psi^{\Delta}(\xi^0).$$

Proof The proof can be seen in the reference [7]. □

Theorem 2.2 *Let the function f be σ_1-completely delta differentiable at the point (t^0, s^0). If the functions φ and ψ have first order partial delta derivative at the point (ξ^0, η^0), then the composite function*

$$F(\xi, \eta) = f(\varphi(\xi, \eta), \psi(\xi, \eta)) \;\; for \;\; (\xi, \eta) \in T_{(1)}xT_{(2)}$$

has the first order partial delta derivatives at (ξ^0, η^0) which are expressed by the formulas

$$\frac{\partial F(\xi^0, \eta^0)}{\Delta_{(1)}\xi} = \frac{\partial f(t^0, s^0)}{\Delta_1 t}\frac{\partial \varphi(\xi^0, \eta^0)}{\Delta_{(1)}\xi} + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s}\frac{\partial \psi(\xi^0, \eta^0)}{\Delta_{(1)}\xi}$$

and

$$\frac{\partial F(\xi^0, \eta^0)}{\Delta_{(2)}\xi} = \frac{\partial f(t^0, s^0)}{\Delta_1 t}\frac{\partial \varphi(\xi^0, \eta^0)}{\Delta_{(2)}\eta} + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s}\frac{\partial \psi(\xi^0, \eta^0)}{\Delta_{(2)}\xi}$$

Proof The proof can be also seen in the reference [7]. □

§3. The directional derivative

Let T be a time scale with the forward jump operator σ and the delta operator Δ. We will assume that $0 \in T$. Further, let $w = (w_1, w_2) \in \mathbf{R}^2$ be a unit vector and let (t^0, s^0) be a fixed point in $\mathbf{R}^2$. Let us set

$$T_1 = \{t = t^0 + \xi w_1 : \; \xi \in T\} \;\; \text{and} \;\; T_2 = \{s = s^0 + \xi w_2 : \; \xi \in T\}.$$

Then T_1 and T_2 are time scales and $t^0 \in T_1, s^0 \in T_2$. Denote the forward jump operators of T_1 and the delta operators by Δ_1.

Definition 3.1 *Let a function $f : T_1xT_2 \to \mathbf{R}$ be given. The directional delta derivative of the function f at the point (t^0, s^0) in the direction of the vector w (along w) is defined as the number*

$$\frac{\partial f(t^0, s^0)}{\Delta w} = F^{\Delta}(0)$$

provided it exists, where

$$F(\xi) = f(t^0 + \xi w_1, s^0 + \xi w_2) \;\; for \;\; \xi \in T.$$

Theorem 3.1 *Suppose that the function f is σ_1-completely delta differentiable at the point (t^0, s^0). Then the directional delta derivative of f at (t^0, s^0) in the direction of the w exists and is expressed by the formula*

$$\frac{\partial f(t^0,s^0)}{\Delta w} = \frac{\partial f(t^0,s^0)}{\Delta_1 t} w_1 + \frac{\partial f(\sigma_1(t^0),s^0)}{\Delta_2 s} w_2.$$

Proof The proof can be found in the reference [7]. □

§4. The tangent vector in Λ^n and some properties

Let us consider the Cartesian product

$$\Lambda^n = T_1 x T_2 x \cdots x T_n = \{P = (x_1, x_2, \cdots, x_n) \text{ for } x_i \in T_i\}$$

where T_i are defined time scale for all $1 \le i \le n,\ n \in \mathbf{N}$. We call Λ^n an n-dimensional Euclidean space on time scale.

Let $x_i : \Lambda^n \to T_i$ be Euclidean coordinate functions on time scale for all $1 \le i \le n,\ n \in \mathbf{N}$, denoted by the set $\{x_1, x_2, \cdots, x_n\}$. Let $f : \Lambda^n \to \Lambda^m$ be a function described by $f(P) = (f_1(P), f_2(P), \cdots, f_m(P))$ at a point $P \in \Lambda^n$. The function f is called σ_1-completely delta differentiable function at the point P provided that, all $f_i,\ i = 1, 2, \cdots, m$ functions are σ_1-completely delta differentiable at the point P. All this kind of functions set will denoted by $\mathbf{C}^{\Delta}_{\sigma_1}$.

Let $P \in \Lambda^n$ and $\{(P, v) = v_P,\ P \in \Lambda^n\}$ be the set of tangent vectors at the point P denoted by $V_P(\Lambda^n)$. Now, we find following properties on this set.

Theorem 4.1 *Let $a, b \in \mathbf{R}$, $f, g \in \mathbf{C}^{\Delta}_{\sigma_1}$ and $v_P, w_P, z_P \in V_P(\Lambda^2)$. Then, the following properties are proven on the directional derivative.*

$$(i) \qquad \frac{\partial f(t^0,s^0)}{\Delta(a v_P + b w_P)} = a\frac{\partial f(t^0,s^0)}{\Delta v_P} + b\frac{\partial f(t^0,s^0)}{\Delta w_P},$$

$$(ii) \qquad \frac{\partial (af+bg)(t^0,s^0)}{\Delta v_P} = a\frac{\partial f(t^0,s^0)}{\Delta v_P} + b\frac{\partial g(t^0,s^0)}{\Delta v_P},$$

$$\begin{aligned}(iii) \qquad \frac{\partial (fg)(t^0,s^0)}{\Delta v_P} &= g(\sigma_1(t^0),s^0)\frac{\partial f(t^0,s^0)}{\Delta v_P} + f(\sigma_1(t^0),\sigma_2(s^0))\frac{\partial g(t^0,s^0)}{\Delta v_P} \\ &-\mu_1(t^0)\frac{\partial f(t^0,s^0)}{\Delta_1 t}\frac{\partial g(t^0,s^0)}{\Delta_1 t}v_1 - \mu_2(s^0)\frac{\partial g(t^0,s^0)}{\Delta_1 t}\frac{\partial f(\sigma_1(t^0),s^0)}{\Delta_2 s}v_1.\end{aligned}$$

Proof Considering Definition 3.1 and Theorem 3.1, we get easily (i) and (ii). Then by Theorem 3.1, we have

$$\begin{aligned}
\frac{\partial(fg)(t^0,s^0)}{\Delta v_P} &= \frac{\partial(fg)(t^0,s^0)}{\Delta_1 t}v_1 + \frac{\partial(fg)(\sigma_1(t^0),s^0)}{\Delta_2 s}v_2 \\
&= \left(\frac{\partial f(t^0,s^0)}{\Delta_1 t}g(t^0,s^0) + f(\sigma_1(t^0),s^0)\frac{\partial g(t^0,s^0)}{\Delta_1 t}\right)v_1 \\
&\quad + \left(\frac{\partial f(\sigma_1(t^0),s^0)}{\Delta_2 s}g(\sigma_1(t^0),s^0) + f(\sigma_1(t^0),\sigma_2(s^0))\frac{\partial g(\sigma_1(t^0),s^0)}{\Delta_2 s}\right)v_2 \\
&= \frac{\partial f(t^0,s^0)}{\Delta_1 t}g(t^0,s^0)v_1 + \frac{\partial f(t^0,s^0)}{\Delta_1 t}g(\sigma_1(t^0),s^0)v_1 \\
&\quad -\frac{\partial f(t^0,s^0)}{\Delta_1 t}g(\sigma_1(t^0),s^0)v_1 + \frac{\partial f(\sigma_1(t^0),s^0)}{\Delta_2 s}g(\sigma_1(t^0),s^0)v_2 \\
&\quad +f(\sigma_1(t^0),s^0)\frac{\partial g(t^0,s^0)}{\Delta_1 t}v_1 + f(\sigma_1(t^0),\sigma_2(s^0))\frac{\partial g(t^0,s^0)}{\Delta_1 t}v_1 \\
&\quad -f(\sigma_1(t^0),\sigma_2(s^0))\frac{\partial g(t^0,s^0)}{\Delta_1 t}v_1 + f(\sigma_1(t^0),\sigma_2(s^0))\frac{\partial g(\sigma_1(t^0),s^0)}{\Delta_2 s}v_2 \\
&= g(\sigma_1(t^0),s^0)\left(\frac{\partial f(t^0,s^0)}{\Delta_1 t}v_1 + \frac{\partial f(\sigma_1(t^0),s^0)}{\Delta_2 s}v_2\right) \\
&\quad +\frac{\partial f(t^0,s^0)}{\Delta_1 t}v_1(g(t^0,s^0) - g(\sigma_1(t^0),s^0)) \\
&\quad +f(\sigma_1(t^0),\sigma_2(s^0))\left(\frac{\partial g(t^0,s^0)}{\Delta_1 t}v_1 + \frac{\partial g(\sigma_1(t^0),s^0)}{\Delta_2 s}v_2\right) \\
&\quad -\frac{\partial g(t^0,s^0)}{\Delta_1 t}v_1(f(\sigma_1(t^0),\sigma_2(s^0)) - f(\sigma_1(t^0),s^0)) \\
&= g(\sigma_1(t^0),s^0)\left(\frac{\partial f(t^0,s^0)}{\Delta v_P}\right) + f(\sigma_1(t^0),\sigma_2(s^0))\left(\frac{\partial g(t^0,s^0)}{\Delta v_P}\right) \\
&\quad -\mu_1(t^0)\frac{\partial f(t^0,s^0)}{\Delta_1 t}\frac{\partial g(t^0,s^0)}{\Delta_1 t}v_1 - \mu_2(s^0)\frac{\partial g(t^0,s^0)}{\Delta_1 t}\frac{\partial f(\sigma_1(t^0),s^0)}{\Delta_2 s}v_1 \qquad \square
\end{aligned}$$

§5. The parameter mapping and delta derivative of vector field along a regular curve

Definition 5.1 *A Δ-regular curve (or an arc of a Δ-regular curve) f is defined as a mapping*

$$x_1 = f_1(t), x_2 = f_2(t), \cdots, x_n = f_n(t),\ t \in [a,b]$$

of the segment $[a,b] \subset T, a < b$, to the space $\mathbf{R}^n$, where $f_1, f_2, \cdots, f_n$ are real-valued functions defined on $[a,b]$ that are Δ-differentiable on $[a,b]^k$ with rd-continuous Δ-derivatives and

$$|f_1^\Delta(t)|^2 + |f_2^\Delta(t)|^2 + \cdots + |f_n^\Delta(t)|^2 \neq 0,\ t \in [a,b]^k.$$

Definition 5.2 *Let $f : T \to \Lambda^n$ be a Δ-differentiable regular curve on $[a,b]^k$. Let T and $\overline{T}$ be a time scales. A parameter mapping $h : \overline{T} \to T$ of the curve f is defined as $t = h(s)$ when h*

and h^{-1} are Δ-differentiable functions.

Thus, according to this new parameter, f can be written as follows:

$$g : \overline{T} \to \Lambda^n, \quad g(s) = f(h(s)).$$

Theorem 5.1 *Let $f : T \to \Lambda^n$ be a Δ-differentiable regular curve. Let the function $h : \overline{T} \to T$ be parameter map of f and be $g = f \circ h$. The Δ-derivative of g is expressed by the formula*

$$g^{\Delta}(s) = \frac{df(h(s))}{\widetilde{\Delta} t} h^{\Delta}(s).$$

Proof Let $f(t) = (f_1(t), f_2(t), \cdots, f_n(t))$ be a regular curve given by the vectorial form in Euclidean space Λ^n. Let h be a parameter mapping of f. Considering chain rule for all f_i, we get

$$\begin{aligned} g^{\Delta}(s) &= \left(\frac{d(f_1 \circ h)}{\Delta(s)}, \frac{d(f_2 \circ h)}{\Delta(s)}, \cdots, \frac{d(f_n \circ h)}{\Delta(s)}\right) \\ &= \left(\frac{df_1(h(s))}{\widetilde{\Delta} t} h^{\Delta}(s), \frac{df_2(h(s))}{\widetilde{\Delta} t} h^{\Delta}(s), \cdots, \frac{df_n(h(s))}{\widetilde{\Delta} t} h^{\Delta}(s)\right) \\ &= \left(\frac{df_1(h(s))}{\widetilde{\Delta} t}, \frac{df_2(h(s))}{\widetilde{\Delta} t}, \cdots, \frac{df_n(h(s))}{\widetilde{\Delta} t}\right) h^{\Delta}(s) \\ &= \frac{df(h(s))}{\widetilde{\Delta} t} h^{\Delta}(s). \qquad \square \end{aligned}$$

Definition 5.3 *A vector field Z is a function which is associated a tangent vector to each point of Λ^n and so $Z(P)$ belongs to the set of tangent vector space $V_P(\Lambda^n)$ at the point P. Generally, a vector field is denoted by*

$$Z(P) = \sum_{i=1}^{n} g_i(t) \frac{\partial}{\partial x_i}|_P$$

where $g_i(t)$ are real valued functions defined on $T = [a, b]$, that are Δ-differentiable on $[a, b]^k$ with rd-continuous Δ-derivative.

Let a function $f : T \to \Lambda^n$, $f(t) = (f_1(t), f_2(t), \cdots, f_n(t))$ be a Δ-differentiable curve and $Z(P)$ be a vector field along it. Thus, we define Δ-derivative of vector fields as follows:

Definition 5.4 *Let $Z(P) = \sum\limits_{i=1}^{n} g_i(t) \frac{\partial}{\partial x_i}|_P$ be a vector field given along the curve f. The Δ-derivative of the function with respect to the parameter is defined as*

$$Z^{\Delta}(P) = \frac{dZ}{\Delta t}|_P = \sum_{i=1}^{n} \lim_{s \to t} \frac{g_i(\sigma(t)) - g_i(s)}{\sigma(t) - s} \frac{\partial}{\partial x_i}|_P = \sum_{i=1}^{n} g_i^{\Delta} \frac{\partial}{\partial x_i}|_P.$$

Theorem 5.2 *Let f be a curve given by $f : T \to \Lambda^n$ and $h : \overline{T} \to T$ be a parameter mapping of f. Let $Z(P) = \sum\limits_{i=1}^{n} g_i(t) \frac{\partial}{\partial x_i}|_{P \in f(t)}$ be a vector field given along the curve f. The Δ-derivative of the function $Z(P)$ according to the new parameter s can be written as:*

$$\left(\frac{dZ}{\Delta s}\right) = \left(\frac{dZ}{\widetilde{\Delta} s}\right)\left(\frac{dh}{\Delta s}\right).$$

Proof When we consider the chain rule on the real valued function for all the Δ-differentiable functions $g_i(t)$, we prove Theorem 5.2. □

Definition 5.5 *Let a vector field* $Z(P) = \sum_{i=1}^{n} g_i(t)\frac{\partial}{\partial x_i}|_{P\in f(t)}$ *be given along a curve* f. *If* $\frac{dZ}{\Delta t} = 0$, $Z(P)$ *is called constant vector field along the curve* f.

Theorem 5.3 *Let* Y *and* Z *be two vector fields along the curve* $f(t)$ *and* $h : T \to \mathbf{R}$ *be a* Δ*-differentiable function. Then,*

(i) $(Y + Z)^{\Delta}(t) = (Y)^{\Delta}(t) + (Z)^{\Delta}(t)$;

(ii) $(hZ)^{\Delta}(t) = (h(\sigma(t)))(Z)^{\Delta}(t) + h^{\Delta}(t)Z(t)$;

(iii) $\langle Y, Z\rangle^{\Delta}(t) = \left\langle Y^{\Delta}(t), Z(\sigma(t))\right\rangle + \left\langle Y(t), Z^{\Delta}(t)\right\rangle$.

where $\langle\ ,\ \rangle$ *is the inner product between the vector fields* Y *and* Z.

Proof The (i) is obvious. Let $Y(t) = \sum_{i=1}^{n} k_i(t)\frac{\partial}{\partial x_i}|_{f(t)}$ and $Z(t) = \sum_{i=1}^{n} g_i(t)\frac{\partial}{\partial x_i}|_{f(t)}$ be two vector fields. Then

$$\begin{aligned}
(hZ)^{\Delta}(t) &= \sum_{i=1}^{n}\lim_{s\to t}\frac{(hg_i)(\sigma(t)) - (hg_i)(s)}{\sigma(t) - s}\frac{\partial}{\partial x_i}|_{f(t)} \\
&= \sum_{i=1}^{n}\lim_{s\to t}\frac{h(\sigma(t))g_i(\sigma(t)) - h(s)g_i(s)}{\sigma(t) - s}\frac{\partial}{\partial x_i}|_{f(t)} \\
&= \sum_{i=1}^{n}\lim_{s\to t}\frac{h(\sigma(t))g_i(\sigma(t)) - h(\sigma(t))g_i(s) + h(\sigma(t))g_i(s) - h(s)g_i(s)}{\sigma(t) - s}\frac{\partial}{\partial x_i}|_{f(t)} \\
&= \sum_{i=1}^{n}\lim_{s\to t}\frac{h(\sigma(t))[g_i(\sigma(t)) - g_i(s)] + [h(\sigma(t)) - h(s)]g_i(s)}{\sigma(t) - s}\frac{\partial}{\partial x_i}|_{f(t)} \\
&= h(\sigma(t))\sum_{i=1}^{n}\lim_{s\to t}\frac{g_i(\sigma(t)) - g_i(s)}{\sigma(t) - s}\frac{\partial}{\partial x_i}|_{f(t)} + \lim_{s\to t}\frac{h(\sigma(t)) - h(s)}{\sigma(t) - s}\sum_{i=1}^{n} g_i(s)\frac{\partial}{\partial x_i}|_{f(t)} \\
&= (h(\sigma(t)))(Z)^{\Delta}(t) + h^{\Delta}(t)Z(t).
\end{aligned}$$

That is the formula (ii). For (iii), we have

$$\begin{aligned}
\langle Y,Z\rangle^{\Delta}(t) &= \sum_{i=1}^{n}\lim_{s\to t}\frac{(k_ig_i)(\sigma(t))-(k_ig_i)(s)}{\sigma(t)-s}\frac{\partial}{\partial x_i}|_{f(t)} \\
&= \sum_{i=1}^{n}\lim_{s\to t}\frac{k_i(\sigma(t))g_i(\sigma(t))-k_i(s)g_i(s)}{\sigma(t)-s}\frac{\partial}{\partial x_i}|_{f(t)} \\
&= \sum_{i=1}^{n}\lim_{s\to t}\frac{k_i(\sigma(t))g_i(\sigma(t))-k_i(s)g_i(\sigma(t))+k_i(s)g_i(\sigma(t))-k_i(s)g_i(s)}{\sigma(t)-s}\frac{\partial}{\partial x_i}|_{f(t)} \\
&= \sum_{i=1}^{n}\lim_{s\to t}\frac{k_i(\sigma(t))g_i(\sigma(t))-k_i(s)g_i(\sigma(t))}{\sigma(t)-s}\frac{\partial}{\partial x_i}|_{f(t)} \\
&\quad +\sum_{i=1}^{n}\lim_{s\to t}\frac{k_i(s)g_i(\sigma(t))-k_i(s)g_i(s)}{\sigma(t)-s}\frac{\partial}{\partial x_i}|_{f(t)} \\
&= \sum_{i=1}^{n}\lim_{s\to t}\frac{k_i(\sigma(t))-k_i(s)}{\sigma(t)-s}g_i(\sigma(t))+\sum_{i=1}^{n}\lim_{s\to t}k_i(s)\lim_{s\to t}\frac{g_i(\sigma(t))-g_i(s)}{\sigma(t)-s} \\
&= \left\langle Y^{\Delta}(t),Z(\sigma(t))\right\rangle+\left\langle Y(t),Z^{\Delta}(t)\right\rangle.
\end{aligned}$$

This completes the proof. □

Definition 5.6 *Let $f:\Lambda^2\to\Lambda^m$, $f(x_1,x_2)=(f_1(x_1,x_2),f_2(x_1,x_2),\cdots,f_n(x_1,x_2))$ be a σ_1-completely delta differentiable function at the point $P(t^0,s^0)\in\Lambda^2$. For any tangent vector $v_p\in V_P(\Lambda^2)$, the Δ-derivative mapping at the point $P(t^0,s^0)$ of f is defined by*

$$f_{*P}^{\Delta}(v_P)=(\frac{\partial f_1(x_1,x_2)}{\Delta v_p},\frac{\partial f_2(x_1,x_2)}{\Delta v_p},\cdots,\frac{\partial f_m(x_1,x_2)}{\Delta v_p})_{f(x_1,x_2)}.$$

$f_{*P}^{\Delta}(v_P)$ is a function from the tangent vector space $V_P(\Lambda^2)$ to tangent vector space $V_P(\Lambda^m)$.

Theorem 5.4 *The function f_{*P} is linear mapping.*

Proof Let us prove the linearity for any $a\in\mathbf{R}$ and for any two tangent vectors $v_P,w_P\in V_P(\Lambda^2)$. In fact,

$$\begin{aligned}
f_{*P}^{\Delta}(av_P+bw_P) &= \left(\frac{\partial f_1(x_1,x_2)}{\Delta(av_p+bw_P)},\frac{\partial f_2(x_1,x_2)}{\Delta(av_p+bw_P)},\cdots,\frac{\partial f_m(x_1,x_2)}{\Delta(av_p+bw_P)}\right)_{f(x_1,x_2)} \\
&= \left(a\frac{\partial f_1(x_1,x_2)}{\Delta v_p}+b\frac{\partial f_1(x_1,x_2)}{\Delta w_P},\cdots,a\frac{\partial f_m(x_1,x_2)}{\Delta v_p}+b\frac{\partial f_m(x_1,x_2)}{\Delta w_P}\right)_{f(x_1,x_2)} \\
&= a\left(\frac{\partial f_1(x_1,x_2)}{\Delta v_p},\cdots,\frac{\partial f_m(x_1,x_2)}{\Delta v_p}\right)_{f(x_1,x_2)} \\
&\quad +b\left(\frac{\partial f_1(x_1,x_2)}{\Delta w_P},\cdots,\frac{\partial f_m(x_1,x_2)}{\Delta w_P}\right)_{f(x_1,x_2)} \\
&= af_{*P}^{\Delta}(v_P)+bf_{*P}^{\Delta}(w_P).
\end{aligned}$$

Thus the proof is completed. □

Definition 5.7 *Let $f(x_1,x_2)=(f_1(x_1,x_2),f_2(x_1,x_2),\cdots,f_n(x_1,x_2))$ be a σ_1-completely delta differentiable function at the point $P(x_1,x_2)\in\Lambda^2$. The Jacobian matrix of f is defined by*

$$J(f,P)=\begin{bmatrix} \frac{\partial f_1(x_1,x_2)}{\partial\Delta_1 x_1} & \frac{\partial f_1(\sigma_1(x_1),x_2)}{\partial\Delta_1 x_1} \\ \frac{\partial f_2(x_1,x_2)}{\partial\Delta_1 x_1} & \frac{\partial f_2(\sigma_1(x_1),x_2)}{\partial\Delta_1 x_1} \\ \cdots & \cdots \\ \frac{\partial f_m(x_1,x_2)}{\partial\Delta_1 x_1} & \frac{\partial f_m(\sigma_1(x_1),x_2)}{\partial\Delta_1 x_1} \end{bmatrix}_{m\times 2}$$

Theorem 5.5 *Let $f:\Lambda^2\rightarrow\Lambda^n$ be a σ_1-completely delta differentiable function. The Δ-derivative mapping for any $P\in\Lambda^2$ and $w_P=\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\in V_P(\Lambda^2)$ is expressed by the formula*

$$f^{\Delta}_{*P}(w_P)=(J(f,P),w_P)^T.$$

Proof Theorem 5.5 is proven considering Definitions 5.6, 5.7 and Theorem 3.1. □

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Basic Properties Of Second Smarandache Bol Loops

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Abstract: The pair $(G_H, \cdot)$ is called a special loop if $(G, \cdot)$ is a loop with an arbitrary subloop $(H, \cdot)$. A special loop $(G_H, \cdot)$ is called a second Smarandache Bol loop($S_{2^{nd}}$BL) if and only if it obeys the second Smarandache Bol identity $(xs \cdot z)s = x(sz \cdot s)$ for all x, z in G and s in H. The popularly known and well studied class of loops called Bol loops fall into this class and so $S_{2^{nd}}$BLs generalize Bol loops. The basic properties of $S_{2^{nd}}$BLs are studied. These properties are all Smarandache in nature. The results in this work generalize the basic properties of Bol loops, found in the Ph.D. thesis of D. A. Robinson. Some questions for further studies are raised.

Key Words: special loop, second Smarandache Bol loop

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§1. Introduction

The study of the Smarandache concept in groupoids was initiated by W. B. Vasantha Kandasamy in [23]. In her book [21] and first paper [22] on Smarandache concept in loops, she defined a Smarandache loop(S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. The present author has contributed to the study of S-quasigroups and S-loops in [5]-[12] by introducing some new concepts immediately after the works of Muktibodh [14]-[15]. His recent monograph [13] gives inter-relationships and connections between and among the various Smarandache concepts and notions that have been developed in the aforementioned papers.

But in the quest of developing the concept of Smarandache quasigroups and loops into a theory of its own just as in quasigroups and loop theory(see [1]-[4], [16], [21]), there is the need to introduce identities for types and varieties of Smarandache quasigroups and loops. For now, a Smarandache loop or Smarandache quasigroup will be called a first Smarandache loop($S_{1^{st}}$-loop) or first Smarandache quasigroup($S_{1^{st}}$-quasigroup).

Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : if $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations ; $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. For each $x \in L$, the elements $x^\rho = xJ_\rho, x^\lambda = xJ_\lambda \in L$ such that $xx^\rho = e^\rho$ and $x^\lambda x = e^\lambda$ are called the right, left inverses of x respectively. Furthermore, if there exists a unique element $e = e_\rho = e_\lambda$ in L called

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the identity element such that for all x in L, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. We write xy instead of $x \cdot y$, and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. For instance, $x \cdot yz$ stands for $x(yz)$. A loop is called a right Bol loop(Bol loop in short) if and only if it obeys the identity

$$(xy \cdot z)y = x(yz \cdot y).$$

This class of loops was the first to catch the attention of loop theorists and the first comprehensive study of this class of loops was carried out by Robinson [18].

The aim of this work is to introduce and study the basic properties of a new class of loops called second Smarandache Bol loops($S_{2^{nd}}$BLs). The popularly known and well studied class of loops called Bol loops fall into this class and so $S_{2^{nd}}$BLs generalize Bol loops. The basic properties of $S_{2^{nd}}$BLs are studied. These properties are all Smarandache in nature. The results in this work generalize the basic properties of Bol loops, found in the Ph.D. thesis [18] and the paper [19] of D. A. Robinson. Some questions for further studies are raised.

§2. Preliminaries

Definition 2.1 *Let $(G, \cdot)$ be a quasigroup with an arbitrary non-trivial subquasigroup $(H, \cdot)$. Then, $(G_H, \cdot)$ is called a special quasigroup with special subquasigroup $(H, \cdot)$. If $(G, \cdot)$ is a loop with an arbitrary non-trivial subloop $(H, \cdot)$. Then, $(G_H, \cdot)$ is called a special loop with special subloop $(H, \cdot)$. If $(H, \cdot)$ is of exponent 2, then $(G_H, \cdot)$ is called a special loop of Smarandache exponent 2.*

A special quasigroup $(G_H, \cdot)$ is called a second Smarandache right Bol quasigroup($S_{2^{nd}}$-right Bol quasigroup) or simply a second Smarandache Bol quasigroup($S_{2^{nd}}$-Bol quasigroup) and abbreviated $S_{2^{nd}}$RBQ or $S_{2^{nd}}$BQ if and only if it obeys the second Smarandache Bol identity($S_{2^{nd}}$-Bol identity) i.e $S_{2^{nd}}$BI

$$(xs \cdot z)s = x(sz \cdot s) \text{ for all } x, z \in G \text{ and } s \in H. \tag{1}$$

Hence, if $(G_H, \cdot)$ is a special loop, and it obeys the $S_{2^{nd}}$BI, it is called a second Smarandache Bol loop($S_{2^{nd}}$-Bol loop) and abbreviated $S_{2^{nd}}$BL.

Remark 2.1 A Smarandache Bol loop(i.e a loop with at least a non-trivial subloop that is a Bol loop) will now be called a first Smarandache Bol loop($S_{1^{st}}$-Bol loop). It is easy to see that a $S_{2^{nd}}$BL is a $S_{1^{nd}}$BL. But the reverse is not generally true. So $S_{2^{nd}}$BLs are particular types of $S_{1^{nd}}$BL. There study can be used to generalise existing results in the theory of Bol loops by simply forcing H to be equal to G.

Definition 2.2 *Let $(G, \cdot)$ be a quasigroup(loop). It is called a right inverse property quasigroup(loop)[RIPQ(RIPL)] if and only if it obeys the right inverse property(RIP) $yx \cdot x^\rho = y$ for all $x, y \in G$. Similarly, it is called a left inverse property quasigroup(loop)[LIPQ(LIPL)] if and*

only if it obeys the left inverse property(LIP) $x^\lambda \cdot xy = y$ *for all* $x, y \in G$. *Hence, it is called an inverse property quasigroup(loop)[IPQ(IPL)] if and only if it obeys both the RIP and LIP.*

$(G, \cdot)$ *is called a right alternative property quasigroup(loop)[RAPQ(RAPL)] if and only if it obeys the right alternative property(RAP)* $y \cdot xx = yx \cdot x$ *for all* $x, y \in G$. *Similarly, it is called a left alternative property quasigroup(loop)[LAPQ(LAPL)] if and only if it obeys the left alternative property(LAP)* $xx \cdot y = x \cdot xy$ *for all* $x, y \in G$. *Hence, it is called an alternative property quasigroup(loop)[APQ(APL)] if and only if it obeys both the RAP and LAP.*

The bijection $L_x : G \to G$ *defined as* $yL_x = x \cdot y$ *for all* $x, y \in G$ *is called a left translation(multiplication) of* G *while the bijection* $R_x : G \to G$ *defined as* $yR_x = y \cdot x$ *for all* $x, y \in G$ *is called a right translation(multiplication) of* G.

$(G, \cdot)$ *is said to be a right power alternative property loop(RPAPL) if and only if it obeys the right power alternative property(RPAP)*

$$xy^n = \underbrace{(((xy)y)y)y \cdots y}_{n\text{-times}} \text{ i.e. } R_{y^n} = R_y^n \text{ for all } x, y \in G \text{ and } n \in \mathbb{Z}.$$

The right nucleus of G *denoted by* $N_\rho(G, \cdot) = N_\rho(G) = \{a \in G : y \cdot xa = yx \cdot a \ \forall\ x, y \in G\}$.

Let $(G_H, \cdot)$ *be a special quasigroup(loop). It is called a second Smarandache right inverse property quasigroup(loop)[*$S_{2^{nd}}$*RIPQ(*$S_{2^{nd}}$*RIPL)] if and only if it obeys the second Smarandache right inverse property(*$S_{2^{nd}}$*RIP)* $ys \cdot s^\rho = y$ *for all* $y \in G$ *and* $s \in H$. *Similarly, it is called a second Smarandache left inverse property quasigroup(loop)[*$S_{2^{nd}}$*LIPQ(*$S_{2^{nd}}$*LIPL)] if and only if it obeys the second Smarandache left inverse property(*$S_{2^{nd}}$*LIP)* $s^\lambda \cdot sy = y$ *for all* $y \in G$ *and* $s \in H$. *Hence, it is called a second Smarandache inverse property quasigroup(loop)[*$S_{2^{nd}}$*IPQ(*$S_{2^{nd}}$*IPL)] if and only if it obeys both the* $S_{2^{nd}}$*RIP and* $S_{2^{nd}}$*LIP.*

$(G_H, \cdot)$ *is called a third Smarandache right inverse property quasigroup(loop)[*$S_{3^{rd}}$*RIPQ(*$S_{3^{rd}}$ *RIPL)] if and only if it obeys the third Smarandache right inverse property(*$S_{3^{rd}}$*RIP)* $sy \cdot y^\rho = s$ *for all* $y \in G$ *and* $s \in H$.

$(G_H, \cdot)$ *is called a second Smarandache right alternative property quasigroup(loop)[*$S_{2^{nd}}$ *RAPQ(*$S_{2^{nd}}$ *RAPL)] if and only if it obeys the second Smarandache right alternative property(*$S_{2^{nd}}$ *RAP)* $y \cdot ss = ys \cdot s$ *for all* $y \in G$ *and* $s \in H$. *Similarly, it is called a second Smarandache left alternative property quasigroup(loop)[*$S_{2^{nd}}$*LAPQ(*$S_{2^{nd}}$ *LAPL)] if and only if it obeys the second Smarandache left alternative property(*$S_{2^{nd}}$ *LAP)* $ss \cdot y = s \cdot sy$ *for all* $y \in G$ *and* $s \in H$. *Hence, it is called an second Smarandache alternative property quasigroup(loop)[*$S_{2^{nd}}$*APQ(*$S_{2^{nd}}$*APL)] if and only if it obeys both the* $S_{2^{nd}}$*RAP and* $S_{2^{nd}}$*LAP.*

$(G_H, \cdot)$ *is said to be a Smarandache right power alternative property loop(SRPAPL) if and only if it obeys the Smarandache right power alternative property(SRPAP)*

$$xs^n = \underbrace{(((xs)s)s)s \cdots s}_{n\text{-times}} \text{ i.e. } R_{s^n} = R_s^n \text{ for all } x \in G,\ s \in H \text{ and } n \in \mathbb{Z}.$$

The Smarandache right nucleus of G_H *denoted by* $SN_\rho(G_H, \cdot) = SN_\rho(G_H) = N_\rho(G) \cap H$. G_H *is called a Smarandache right nuclear square special loop if and only if* $s^2 \in SN_\rho(G_H)$ *for all* $s \in H$.

Remark 2.2 A Smarandache; RIPQ or LIPQ or IPQ(i.e a loop with at least a non-trivial subquasigroup that is a RIPQ or LIPQ or IPQ) will now be called a first Smarandache; RIPQ or LIPQ or IPQ($S_{1^{st}}$RIPQ or $S_{1^{st}}$LIPQ or $S_{1^{st}}$IPQ). It is easy to see that a $S_{2^{st}}$RIPQ or $S_{2^{st}}$LIPQ or $S_{2^{st}}$IPQ is a $S_{1^{st}}$RIPQ or $S_{1^{st}}$LIPQ or $S_{1^{st}}$IPQ respectively. But the reverse is not generally true.

Definition 2.3 *Let $(G,\cdot)$ be a quasigroup(loop). The set $SYM(G,\cdot) = SYM(G)$ of all bijections in G forms a group called the permutation(symmetric) group of G. The triple (U,V,W) such that $U,V,W \in SYM(G,\cdot)$ is called an autotopism of G if and only if*

$$xU \cdot yV = (x \cdot y)W \ \forall \ x,y \in G.$$

The group of autotopisms of G is denoted by $AUT(G,\cdot) = AUT(G)$.

Let $(G_H,\cdot)$ be a special quasigroup(loop). The set $SSYM(G_H,\cdot) = SSYM(G_H)$ of all Smarandache bijections(S-bijections) in G_H i.e $A \in SYM(G_H)$ such that $A : H \to H$ forms a group called the Smarandache permutation(symmetric) group[S-permutation group] of G_H. The triple (U,V,W) such that $U,V,W \in SSYM(G_H,\cdot)$ is called a first Smarandache autotopism($S_{1^{st}}$ autotopism) of G_H if and only if

$$xU \cdot yV = (x \cdot y)W \ \forall \ x,y \in G_H.$$

If their set forms a group under componentwise multiplication, it is called the first Smarandache autotopism group($S_{1^{st}}$ autotopism group) of G_H and is denoted by $S_{1^{st}}AUT(G_H,\cdot) = S_{1^{st}}AUT(G_H)$.

The triple (U,V,W) such that $U,W \in SYM(G,\cdot)$ and $V \in SSYM(G_H,\cdot)$ is called a second right Smarandache autotopism($S_{2^{nd}}$ right autotopism) of G_H if and only if

$$xU \cdot sV = (x \cdot s)W \ \forall \ x \in G \textit{ and } s \in H.$$

If their set forms a group under componentwise multiplication, it is called the second right Smarandache autotopism group($S_{2^{nd}}$ right autotopism group) of G_H and is denoted by $S_{2^{nd}}RAUT(G_H,\cdot) = S_{2^{nd}}RAUT(G_H)$.

The triple (U,V,W) such that $V,W \in SYM(G,\cdot)$ and $U \in SSYM(G_H,\cdot)$ is called a second left Smarandache autotopism($S_{2^{nd}}$ left autotopism) of G_H if and only if

$$sU \cdot yV = (s \cdot y)W \ \forall \ y \in G \textit{ and } s \in H.$$

If their set forms a group under componentwise multiplication, it is called the second left Smarandache autotopism group($S_{2^{nd}}$ left autotopism group) of G_H and is denoted by

$$S_{2^{nd}}LAUT(G_H,\cdot) = S_{2^{nd}}LAUT(G_H).$$

Let $(G_H,\cdot)$ be a special quasigroup(loop) with identity element e. A mapping $T \in SSYM(G_H)$ is called a first Smarandache semi-automorphism($S_{1^{st}}$ semi-automorphism) if and only if $eT = e$ and

$$(xy \cdot x)T = (xT \cdot yT)xT \text{ for all } x, y \in G.$$

A mapping $T \in SSYM(G_H)$ is called a second Smarandache semi-automorphism($S_{2^{nd}}$ semi-automorphism) if and only if $eT = e$ and

$$(sy \cdot s)T = (sT \cdot yT)sT \text{ for all } y \in G \text{ and all } s \in H.$$

A special loop $(G_H, \cdot)$ is called a first Smarandache semi-automorphism inverse property loop($S_{1^{st}}$SAIPL) if and only if J_ρ is a $S_{1^{st}}$ semi-automorphism.

A special loop $(G_H, \cdot)$ is called a second Smarandache semi-automorphism inverse property loop($S_{2^{nd}}$SAIPL) if and only if J_ρ is a $S_{2^{nd}}$ semi-automorphism.

Let $(G_H, \cdot)$ be a special quasigroup(loop). A mapping $A \in SSYM(G_H)$ is a

1. first Smarandache pseudo-automorphism($S_{1^{st}}$ pseudo-automorphism) of G_H if and only if there exists a $c \in H$ such that $(A, AR_c, AR_c) \in S_{1^{st}}AUT(G_H)$. c is reffered to as the first Smarandache companion($S_{1^{st}}$ companion) of A. The set of such As' is denoted by $S_{1^{st}}PAUT(G_H, \cdot) = S_{1^{st}}PAUT(G_H)$.

2. second right Smarandache pseudo-automorphism($S_{2^{nd}}$ right pseudo-automorphism) of G_H if and only if there exists a $c \in H$ such that $(A, AR_c, AR_c) \in S_{2^{nd}}RAUT(G_H)$. c is reffered to as the second right Smarandache companion($S_{2^{nd}}$ right companion) of A. The set of such As' is denoted by $S_{2^{nd}}RPAUT(G_H, \cdot) = S_{2^{nd}}RPAUT(G_H)$.

3. second left Smarandache pseudo-automorphism($S_{2^{nd}}$ left pseudo-automorphism) of G_H if and only if there exists a $c \in H$ such that $(A, AR_c, AR_c) \in S_{2^{nd}}LAUT(G_H)$. c is reffered to as the second left Smarandache companion($S_{2^{nd}}$ left companion) of A. The set of such As' is denoted by $S_{2^{nd}}LPAUT(G_H, \cdot) = S_{2^{nd}}LPAUT(G_H)$.

§3. Main Results

Theorem 3.1 *Let the special loop $(G_H, \cdot)$ be a $S_{2^{nd}}$BL. Then it is both a $S_{2^{nd}}$RIPL and a $S_{2^{nd}}$RAPL.*

Proof

1. In the $S_{2^{nd}}$BI, substitute $z = s^\rho$, then $(xs \cdot s^\rho)s = x(ss^\rho \cdot s) = xs$ for all $x \in G$ and $s \in H$. Hence, $xs \cdot s^\rho = x$ which is the $S_{2^{nd}}$RIP.

2. In the $S_{2^{nd}}$BI, substitute $z = e$ and get $xs \cdot s = x \cdot ss$ for all $x \in G$ and $s \in H$. Which is the $S_{2^{nd}}$RAP. □

Remark 3.1 Following Theorem 3.1, we know that if a special loop $(G_H, \cdot)$ is a $S_{2^{nd}}$BL, then its special subloop $(H, \cdot)$ is a Bol loop. Hence, $s^{-1} = s^\lambda = s^\rho$ for all $s \in H$. So, if $n \in \mathbb{Z}^+$, define xs^n recursively by $s^0 = e$ and $s^n = s^{n-1} \cdot s$. For any $n \in \mathbb{Z}^-$, define s^n by $s^n = (s^{-1})^{|n|}$.

Theorem 3.2 *If $(G_H, \cdot)$ is a $S_{2^{nd}}$BL, then*

$$xs^n = xs^{n-1} \cdot s = xs \cdot s^{n-1} \tag{2}$$

for all $x \in G$, $s \in H$ *and* $n \in \mathbb{Z}$.

it Proof Trivially, (2) holds for $n = 0$ and $n = 1$. Now assume for $k > 1$,

$$xs^k = xs^{k-1} \cdot s = xs \cdot s^{k-1} \tag{3}$$

for all $x \in G$, $s \in H$. In particular, $s^k = s^{k-1} \cdot s = s \cdot s^{k-1}$ for all $s \in H$. So, $xs^{k+1} = x \cdot s^k s = x(ss^{k-1} \cdot s) = (xs \cdot s^{k-1})s = xs^k \cdot s$ for all $x \in G$, $s \in H$. Then, replacing x by xs in (3), $xs \cdot s^k = (xs \cdot s^{k-1})s = x(ss^{k-1} \cdot s) = x(s^{k-1}s \cdot s) = x \cdot s^k s = xs^{k+1}$ for all $x \in G$, $s \in H$.(Note that the $\mathrm{S}_{2^{\mathrm{nd}}}$BI has been used twice.)

Thus, (2) holds for all integers $n \geqslant 0$.

Now, for all integers $n > 0$ and all $x \in G$, $s \in H$, applying (2) to x and s^{-1} gives $x(s^{-1})^{n+1} = x(s^{-1})^n \cdot s^{-1} = xs^{-n} \cdot s^{-1}$, and (2) applied to xs and s^{-1} gives $xs \cdot (s^{-1})^{n+1} = (xs \cdot s^{-1})(s^{-1})^n = xs^{-n}$. Hence, $xs^{-n} = xs^{-n-1} \cdot s = xs \cdot s^{-n-1}$ and the proof is complete.(Note that the $\mathrm{S}_{2^{\mathrm{nd}}}$RIP of Theorem 3.1 has been used.) □

Theorem 3.3 *If* $(G_H, \cdot)$ *is a* $S_{2^{nd}}BL$*, then*

$$xs^m \cdot s^n = xs^{m+n} \tag{4}$$

for all $x \in G$, $s \in H$ *and* $m, n \in \mathbb{Z}$.

Proof The desired result clearly holds for $n = 0$ and by Theorem 3.2, it also holds for $n = 1$.

For any integer $n > 1$, assume that (4) holds for all $m \in \mathbb{Z}$ and all $x \in G$, $s \in H$. Then, using Theorem 3.2, $xs^{m+n+1} = xs^{m+n} \cdot s = (xs^m \cdot s^n)s = xs^m \cdot s^{n+1}$ for all $x \in G$, $s \in H$ and $m \in \mathbb{Z}$. So, (4) holds for all $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Recall that $(s^n)^{-1} = s^{-n}$ for all $n \in \mathbb{Z}^+$ and $s \in H$. Replacing m by $m - n$, $xs^{m-n} \cdot s^n = xs^m$ and, hence, $xs^{m-n} = xs^m \cdot (s^n)^{-1} = xs^m \cdot s^{-n}$ for all $m \in \mathbb{Z}$ and $x \in G$, $s \in H$. □

Corollary 3.1 *Every* $S_{2^{nd}}BL$ *is a SRPAPL.*

Proof When $n = 1$, the SRPAP is true. When $n = 2$, the SRPAP is the SRAP. Let the SRPAP be true for $k \in \mathbb{Z}^+$; $R_{s^k} = R_s^k$ for all $s \in H$. Then, by Theorem **??**, $R_s^{k+1} = R_s^k R_s = R_{s^k} R_s = R_{s^{k+1}}$ for all $s \in H$. □

Lemma 3.1 *Let* $(G_H, \cdot)$ *be a special loop. Then,* $S_{1^{st}}AUT(G_H, \cdot) \leqslant AUT(G_H, \cdot)$, $S_{2^{nd}}RAUT(G_H, \cdot) \leqslant AUT(H, \cdot)$ *and* $S_{2^{nd}}LAUT(G_H, \cdot) \leqslant AUT(H, \cdot)$*. But,* $S_{2^{nd}}RAUT(G_H, \cdot) \nleqslant AUT(G_H, \cdot)$ *and* $S_{2^{nd}}LAUT(G_H, \cdot) \nleqslant AUT(G_H, \cdot)$.

Proof These are easily proved by using the definitions of the sets relative to componentwise multiplication. □

Lemma 3.2 *Let $(G_H, \cdot)$ be a special loop. Then, $S_{2^{nd}}RAUT(G_H, \cdot)$ and $S_{2^{nd}}LAUT(G_H, \cdot)$ are groups under componentwise multiplication.*

Proof These are easily proved by using the definitions of the sets relative to componentwise multiplication. □

Lemma 3.3 *Let $(G_H, \cdot)$ be a special loop.*

(1) If $(U, V, W) \in S_{2^{nd}}RAUT(G_H, \cdot)$ and G_H has the $S_{2^{nd}}RIP$, then

$$(W, J_\rho V J_\rho, U) \in S_{2^{nd}}RAUT(G_H, \cdot).$$

(2) If $(U, V, W) \in S_{2^{nd}}LAUT(G_H, \cdot)$ and G_H has the $S_{2^{nd}}LIP$, then

$$(J_\lambda U, W, V) \in S_{2^{nd}}LAUT(G_H, \cdot).$$

Proof (1)$(U, V, W) \in \mathrm{S}_{2^{\mathrm{nd}}}RAUT(G_H, \cdot)$ implies that $xU \cdot sV = (x \cdot s)W$ for all $x \in G$ and $s \in H$. So, $(xU \cdot sV)(sV)^\rho = (x \cdot s)W \cdot (sV)^\rho \Rightarrow xU = (xs^\rho)W \cdot (s^\rho V)^\rho \Rightarrow (xs)U = (xs \cdot s^\rho)W \cdot (s^\rho V)^\rho \Rightarrow (xs)U = xW \cdot sJ_\rho V J_\rho \Rightarrow (W, J_\rho V J_\rho, U) \in \mathrm{S}_{2^{\mathrm{nd}}}RAUT(G_H, \cdot)$.

(2)$(U, V, W) \in \mathrm{S}_{2^{\mathrm{nd}}}LAUT(G_H, \cdot)$ implies that $sU \cdot xV = (s \cdot x)W$ for all $x \in G$ and $s \in H$. So, $(sU)^\lambda \cdot (sU \cdot xV) = (sU)^\lambda \cdot (s \cdot x)W \Rightarrow xV = (sU)^\lambda \cdot (sx)W \Rightarrow xV = (s^\lambda U)^\lambda \cdot (s^\lambda x)W \Rightarrow (sx)V = (s^\lambda U)^\lambda \cdot (s^\lambda \cdot sx)W \Rightarrow (sx)V = sJ_\lambda U J_\lambda \cdot xW \Rightarrow (J_\lambda U, W, V) \in \mathrm{S}_{2^{\mathrm{nd}}}LAUT(G_H, \cdot)$. □

Theorem 3.4 *Let $(G_H, \cdot)$ be a special loop. $(G_H, \cdot)$ is a $S_{2^{nd}}BL$ if and only if $(R_s^{-1}, L_sR_s, R_s) \in S_{1^{st}}AUT(G_H, \cdot)$.*

Proof G_H is a $\mathrm{S}_{2^{\mathrm{nd}}}$BL iff $(xs \cdot z)s = x(sz \cdot s)$ for all $x, z \in G$ and $s \in H$ iff $(xR_s \cdot z)R_s = x(zL_sR_s)$ iff $(xz)R_s = xR_s^{-1} \cdot zL_sR_s$ iff $(R_s^{-1}, L_sR_s, R_s) \in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H, \cdot)$. □

Theorem 3.5 *Let $(G_H, \cdot)$ be a $S_{2^{nd}}BL$. G_H is a $S_{2^{nd}}SAIPL$ if and only if G_H is a $S_{3^{rd}}RIPL$.*

Proof Keeping the $\mathrm{S}_{2^{\mathrm{nd}}}$BI and the $\mathrm{S}_{2^{\mathrm{nd}}}$RIP in mind, it will be observed that if G_H is a $\mathrm{S}_{3^{\mathrm{rd}}}$RIPL, then $(sy \cdot s)(s^\rho y^\rho \cdot s^\rho) = [((sy \cdot s)s^\rho)y^\rho]s^\rho = (sy \cdot y^\rho)s^\rho = ss^\rho = e$. So, $(sy \cdot s)^\rho = s^\rho y^\rho \cdot s^\rho$. The proof of the necessary part follows by the reverse process. □

Theorem 3.6 *Let $(G_H, \cdot)$ be a $S_{2^{nd}}BL$. If $(U, T, U) \in S_{1^{st}}AUT(G_H, \cdot)$. Then, T is a $S_{2^{nd}}$ semi-automorphism.*

Proof If $(U, T, U) \in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H, \cdot)$, then, $(U, T, U) \in \mathrm{S}_{2^{\mathrm{nd}}}RAUT(G_H, \cdot) \cap \mathrm{S}_{2^{\mathrm{nd}}}LAUT(G_H, \cdot)$.

Let $(U, T, U) \in \mathrm{S}_{2^{\mathrm{nd}}}RAUT(G_H, \cdot)$, then $xU \cdot sT = (xs)U$ for all $x \in G$ and $s \in H$. Set $s = e$, then $eT = e$. Let $u = eU$, then $u \in H$ since $(U, T, U) \in \mathrm{S}_{2^{\mathrm{nd}}}LAUT(G_H, \cdot)$. For $x = e$, $U = TL_u$. So, $xTL_u \cdot sT = (xs)TL_u$ for all $x \in G$ and $s \in H$. Thus,

$$(u \cdot xT) \cdot sT = u \cdot (xs)T. \tag{5}$$

Replace x by sx in (5), to get

$$[u \cdot (sx)T] \cdot sT = u \cdot (sx \cdot s)T. \qquad (6)$$

$(U,T,U) \in S_{2^{\rm nd}}LAUT(G_H,\cdot)$ implies that $sU \cdot xT = (sx)U$ for all $x \in G$ and $s \in H$ implies $sTL_u \cdot xT = (sx)TL_u$ implies $(u \cdot sT) \cdot xT = u \cdot (sx)T$. Using this in (6) gives $[(u \cdot sT) \cdot xT] \cdot sT = u \cdot (sx \cdot s)T$. By the $S_{2^{\rm nd}}$BI, $u[(sT \cdot xT) \cdot sT] = u \cdot (sx \cdot s)T \Rightarrow (sT \cdot xT) \cdot sT = (sx \cdot s)T$. □

Corollary 3.2 *Let $(G_H,\cdot)$ be a $S_{2^{nd}}BL$ that is a Smarandache right nuclear square special loop. Then, $L_sR_s^{-1}$ is a $S_{2^{nd}}$ semi-automorphism.*

Proof $s^2 \in SN_\rho(G_H)$ for all $s \in H$ iff $xy \cdot s^2 = x \cdot ys^2$ iff $(xy)R_{s^2} = x \cdot yR_{s^2}$ iff $(xy)R_s^2 = x \cdot yR_s^2(\because$ of $S_{2^{\rm nd}}$RAP) iff $(I,R_s^2,R_s^2) \in S_{1^{\rm st}}AUT(G_H,\cdot)$ iff $(I,R_s^{-2},R_s^{-2}) \in S_{1^{\rm st}}AUT(G_H,\cdot)$. Recall from Theorem 3.4 that, $(R_s^{-1},L_sR_s,R_s) \in S_{1^{\rm st}}AUT(G_H,\cdot)$. So, $(R_s^{-1},L_sR_s,R_s)(I,R_s^{-2},R_s^{-2}) = (R_s^{-1},L_sR_s^{-1},R_s^{-1}) \in S_{1^{\rm st}}AUT(G_H,\cdot) \Rightarrow L_sR_s^{-1}$ is a $S_{2^{\rm nd}}$ semi-automorphism by Theorem 3.6. □

Corollary 3.3 *If a $S_{2^{nd}}BL$ is of Smarandache exponent 2, then, $L_sR_s^{-1}$ is a $S_{2^{nd}}$ semi-automorphism.*

Proof These follows from Theorem 3.2. □

Theorem 3.7 *Let $(G_H,\cdot)$ be a $S_{2^{nd}}BL$. Let $(U,V,W) \in S_{1^{st}}AUT(G_H,\cdot)$, $s_1 = eU$ and $s_2 = eV$. Then, $A = UR_s^{-1} \in S_{1^{st}}PAUT(G_H)$ with $S_{1^{st}}$ companion $c = s_1s_2 \cdot s_1$ such that $(U,V,W) = (A,AR_c,AR_c)(R_s^{-1},L_sR_s,R_s)^{-1}$.*

Proof By Theorem 3.4, $(R_s^{-1},L_sR_s,R_s) \in S_{1^{\rm st}}AUT(G_H,\cdot)$ for all $s \in H$. Hence, $(A,B,C) = (U,V,W)(R_{s_1}^{-1},L_{s_1}R_{s_1},R_{s_1}) = (UR_{s_1}^{-1},VL_{s_1}R_{s_1},WR_{s_1}) \in S_{1^{\rm st}}AUT(G_H,\cdot) \Rightarrow A = UR_{s_1}^{-1}$, $B = VL_{s_1}R_{s_1}$ and $C = WR_{s_1}$. That is, $aA \cdot bB = (ab)C$ for all $a,b \in G_H$. Since $eA = e$, then setting $a = e$, $B = C$. Then for $b = e$, $B = AR_{eB}$. But $eB = eVL_{s_1}R_{s_1} = s_1s_2 \cdot s_1$. Thus, $(A,AR_{eB},AR_{eB}) \in S_{1^{\rm st}}AUT(G_H,\cdot) \Rightarrow A \in S_{1^{\rm st}}PAUT(G_H,\cdot)$ with $S_{1^{\rm st}}$ companion $c = s_1s_2 \cdot s_1 \in H$. □

Theorem 3.8 *Let $(G_H,\cdot)$ be a $S_{2^{nd}}BL$. Let $(U,V,W) \in S_{2^{nd}}LAUT(G_H,\cdot) \cap S_{2^{nd}}RAUT(G_H,\cdot)$, $s_1 = eU$ and $s_2 = eV$. Then, $A = UR_s^{-1} \in S_{2^{nd}}LPAUT(G_H) \cap S_{2^{nd}}RPAUT(G_H)$ with $S_{2^{nd}}$ left companion and $S_{2^{nd}}$ right companion $c = s_1s_2 \cdot s_1$ such that*

$$(U,V,W) = (A,AR_c,AR_c)(R_s^{-1},L_sR_s,R_s)^{-1}.$$

Proof The proof of this is very similar to the proof of Theorem 3.7. □

Remark 3.2 Every Bol loop is a $S_{2^{\rm nd}}$BL. Most of the results on basic properties of Bol loops in Chapter 2 of [18] can easily be deduced from the results in this paper by simply forcing H to be equal to G.

Question 3.1 *Let $(G_H,\cdot)$ be a special quasigroup(loop). Are the sets*

$$S_{1^{st}}PAUT(G_H),\ S_{2^{nd}}RPAUT(G_H)\ \text{and}\ S_{2^{nd}}LPAUT(G_H)$$

groups under mapping composition?

Question 3.2 *Let $(G_H, \cdot)$ be a special quasigroup(loop). Can we find a general method(i.e not an acceptable $S_{2^{nd}}BL$ with carrier set $\mathbb{N}$) of constructing a $S_{2^{nd}}BL$ that is not a Bol loop just like Robinson [18], Solarin and Sharma [20] were able to use general methods to construct Bol loops.*

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Smarandachely Precontinuous maps and Preopen Sets in Topological Vector Spaces

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Abstract: It is shown that linear functional on topological vector spaces are Smarandachely precontinuous. Prebounded, totally prebounded and precompact sets in topological vector spaces are identified.

Key Words: Smarandachely Preopen set, precompact set, Smarandachely precontinuous map.

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§1. Introduction

N. Levine [7] introduced the theory of semi-open sets and the theory of α-sets for topological spaces. For a systematic development of semi-open sets and the theory of α-sets one may refer to [1], [2], [4], [5] and [9]. The notion of preopen sets for topological spaces was introduced by S. N. Mashour, M. E. Abd El-Moncef and S.N. El-Deep in [8]. These concepts above are closely related. It is known that, in a topological space, a set is preopen and semi-open if and only if it is an α-set [10], [11]. Our object in section 3 is to define a prebounded set, totally prebounded set, and precompact set in a topological vector space. In Sections 3 and 4 we identify them. Moreover, in Section 2, we show that every linear functional on a topological vector space is precontinuous and deduce that every topological vector space is a prehausdorff space.

§2. Precontinuous maps

We recall the following definitions [2], [8].

Definition 2.1 *Let X be a topological space. A subset S of X is said to be Smarandachely preopen if there exists a set $U \subset \mathrm{cl}(S)$ such that $S \subset \mathrm{int}(\mathrm{cl}(S) \cup U)$. A Smarandachely preneighbourhood of the point $x \in X$ is any Smarandachely preopen set containing x. Particularly, a*

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Smarandachely $\varnothing$-preopen set S is usually called a preopen set.

Definition 2.2 *Let X and Y be topological spaces and $f : X \to Y$. The function f is said to be Smarandachely precontinuous if the inverse image $f^{-1}(B)$ of each open set B in Y is a Smarandachely preopen set in X. The function f is said to be Smarandachely preopen if the image $f(A)$ of every open set A in X is Smarandachely preopen in Y. Particularly, if we replace each Smarandachely preopen by preopen, f is called to be precontinuous.*

The following lemma is obvious.

Lemma 2.1 *Let X and Y be topological vector spaces and $f : X \to Y$ linear. The function f is preopen if and only if, for every open set U containing $0 \in X$, $0 \in Y$ is an interior point of* $\mathrm{cl}(f(U))$.

The following two theorems are known but we include the proofs for convenience of the reader.

Theorem 2.1 *Let X, Y be topological vector spaces and let Y have the Baire property, that is, whenever $Y = \cup_{n=1}^{\infty} B_n$ with closed sets B_n, there is is N such that $\mathrm{int}(B_N)$ is nonempty. Let $f : X \to Y$ be linear and $f(X) = Y$. Then f is preopen.*

Proof Let $U \subset X$ be a neighborhood of 0. There is a neighborhood V of 0 such that $V - V \subset U$. Since V is a neighborhood of 0 we have $X = \cup_{n=1}^{\infty} nV$. It follows from linearity and surjectivity of f that $Y = \cup_{n=1}^{\infty} nf(V)$. Since Y has the Baire property, there is N such that $\mathrm{cl}(Nf(V)) = N\mathrm{cl}(f(V))$ contains an open set S which is not empty. Then $\mathrm{cl}(f(V))$ contains the open set $T = \frac{1}{N}S$. It follows that

$$T - T \subset \mathrm{cl}(f(V)) - \mathrm{cl}(f(V)) \subset \mathrm{cl}(f(V) - f(V)) = \mathrm{cl}(f(V - V)) \subset \mathrm{cl}(f(U)).$$

The set $T - T$ is open and contains 0. Therefore, $0 \in Y$ is an interior point of $\mathrm{cl}(f(U))$. From Lemma 2.1 we conclude that f is preopen. $\square$

Note that f can be any linear surjective map. It is not necessary to assume that f is continuous or precontinuous.

Theorem 2.2 *Let X, Y be topological vector spaces, and let X have the Baire property. Then every linear map $f : X \to Y$ is precontinuous.*

Proof Let $G = \{(x, f(x)) : x \in X\}$ be the graph of f. The projections $\pi_1 : G \to X$ and $\pi_2 : G \to Y$ are continuous. The projection $\pi_1 : G \to X$ is bijective. It follows from Theorem **??** that π_1 is preopen. Therefore, the inverse mapping π_1^{-1} is precontinuous. Then $f = \pi_2 \circ \pi_1^{-1}$ is precontinuous. $\square$

Theorem 2.2 shows that many linear maps are automatically precontinuous. Therefore, it is natural to ask for an example of a linear map which is not precontinuous.

Let $X = C[0, 1]$ be the vector space of real-valued continuous functions on $[0, 1]$ equipped with the norm

$$\|f\|_1 = \int_0^1 |f(x)|\, dx.$$

Let $Y = C[0,1]$ be equipped with the norm

$$\|f\|_\infty = \max_{x\in[0,1]} |f(x)|.$$

Lemma 2.2 *The identity operator $T : X \to Y$ is not precontinuous.*

Proof Let $U = \{f \in C[0,1] : \|f\|_\infty < 1\}$ which is an open subset of Y. Let $\mathrm{cl}(U)$ be the closure of U in X. We claim that

$$\text{(2.1)} \qquad \mathrm{cl}(U) \subset \{f \in C[0,1] : \|f\|_\infty \leqslant 1\}.$$

For the proof, consider a sequence $f_n \in U$ and a function $f \in C[0,1]$ such that $\{f_n\}$ converges to f in X. Suppose that there is $x_0 \in [0,1]$ such that $f(x_0) > 1$. By continuity of f, there are $a < b$ and $\delta > 0$ such that $0 \leqslant a \leqslant x_0 \leqslant b \leqslant 1$ and $f(x) > 1+\delta$ for $x \in (a,b)$. Then, as $n \to \infty$,

$$(b-a)\delta \leqslant \int_a^b |f_n(x) - f(x)|\, dx \leqslant \int_0^1 |f_n(x) - f(x)|\, dx \to 0$$

which is a contradiction. Therefore, $f(x) \leqslant 1$ for all $x \in [0,1]$. Similarly, we show that $f(x) \geqslant -1$ for all $x \in [0,1]$. Now $0 \in U = T^{-1}(U)$ but U is not preopen in X. We see this as follows. Suppose that U is preopen in X. The sequence $g_n(x) = 2x^n$ converges to 0 in X. Therefore, $g_n \in \mathrm{cl}(U)$ for some n and (2.1) implies $2 = \|g_n\|_\infty \leqslant 1$ which is a contradiction. □

We can improve Theorem 2.2 for linear functionals.

Theorem 2.3 *Let f be a linear functional on a topological vector space X. If V is a preopen subset of $\mathbb{R}$ then $f^{-1}(V)$ is a preopen subset of X. In particular, f is precontinuous.*

Proof We distinguish the cases that f is continuous or discontinuous.

Suppose that f is continuous. If $f(x) = 0$ for all $x \in X$ the statement of the theorem is true. Suppose that f is onto. We choose $u \in X$ such that $f(u) = 1$. Let V be a preopen subset of $\mathbb{R}$, and set $U := f^{-1}(V)$. Let $x \in U$ so $f(x) \in V$. Since V is preopen, there is $\delta > 0$ such that

$$\text{(2.2)} \qquad I := (f(x) - \delta, f(x) + \delta) \subset \mathrm{cl}(V).$$

Since f is continuous, $f^{-1}(I)$ is an open subset of X containing x. We claim that

$$\text{(2.3)} \qquad f^{-1}(I) \subset \mathrm{cl}(U).$$

In order to prove (2.3), let $y \in f^{-1}(I)$ so $f(y) \in I$. By (2.2), there is a sequence $\{t_n\}$ in V converging to $f(y)$. Set

$$y_n := y + (t_n - f(y))u.$$

We have $f(y_n) = t_n \in V$ so $y_n \in U$. Since X is a topological vector space, y_n converges to y. This establishes (2.3). It follows that U is preopen.

Suppose now that f is not continuous. By [3, Corollary 22.1], $N(f) = \{x \in X : f(x) = 0\}$ is not closed. Therefore, there is $y \in \mathrm{cl}(N(f))$ such that $y \notin N(f)$ so $f(y) \neq 0$. Let x be any vector in X. There is $t \in \mathbb{R}$ such that $f(x) = tf(y)$ and so $x - ty \in N(f)$. It follows that $x \in \mathrm{cl}(N(f))$. We have shown that $N(f)$ is dense in X. Let $a \in \mathbb{R}$. There is $y \in X$ such that $f(y) = a$. Then $f^{-1}(\{a\}) = y + N(f)$ and so the closure of $f^{-1}(\{a\})$ is $y + \mathrm{cl}(N(f)) = X$. Therefore, $f^{-1}(\{a\})$ is dense for every $a \in \mathbb{R}$. Let V be a preopen set in $\mathbb{R}$. If V is empty then $f^{-1}(V)$ is empty and so is preopen. If V is not empty choose $a \in V$. Then $f^{-1}(V) \supset f^{-1}(\{a\})$ and so $f^{-1}(V)$ is dense. Therefore, $f^{-1}(V)$ is preopen. □

§3. Subsets of topological vector spaces

In this section our principal goal is to define prebounded sets, totally prebounded sets and precompact sets in a topological vector space, and to find relations between them. We begin this section with some definitions.

Definition 3.1 *A subset E of a topological vector space X is said to be prebounded if for every preneighbourhood V of 0 there exists $s > 0$ such that $E \subset tV$ for all $t > s$.*

Definition 3.2 *A subset E of a topological vector space X is said to be totally prebounded if for every preneighbourhood U of 0 there exists a finite subset F of X such that $E \subset F + U$.*

Definition 3.3 *A subset E of a topological vector space X is said to be precompact if every preopen cover of E admits a finite subcover.*

Lemma 3.1 *Every precompact set in a topological vector space X is totally prebounded.*

Proof Let E be precompact. Let V be preopen with $0 \in V$. Then the collection $\{x + V : x \in E\}$ is a cover of E consisting of preopen sets. There are $x_1, x_2, \ldots, x_n \in E$ such that $E \subset \bigcup_{i=1}^{n} \{x_i + V\}$. Therefore, E is totally prebounded. □

Lemma 3.2 *In a topological vector space X the singleton $\{0\}$ is the only prebounded set.*

Proof It is enough to show that every singleton $\{u\}$, $u \neq 0$, is not prebounded. Let $V = X - \left\{\frac{1}{n}u : n \in \mathbb{N}\right\}$. The closure of V is X so V is preopen . But $\{u\}$ is not subset of nV for $n = 1, 2, 3, \ldots$ Therefore, $\{u\}$ is not prebounded. □

Theorem 3.1 *If E is a prebounded subset of a topological vector space X, then E is totally prebounded. The converse statement is not true.*

Proof This follows from the fact that every finite set is totally prebounded, and by using Lemma 3.2. □

§4. **Applications of Theorem** 2.3

We need the following known lemma.

Lemma 4.1 *If U, V are two vector spaces, and W is a linear subspace of U and $f : W \to V$ is a linear map. then there is a linear map $g : U \to V$ such that $f(x) = g(x)$ for all $x \in W$.*

Proof We choose a basis A in W and then extend to a basis $B \supset A$ in U. We define $h(a) = f(a)$ for $a \in A$ and $h(b)$ arbitrary in V for $b \in B - A$. There is a unique linear map $g : U \to V$ such that $g(b) = h(b)$ for $b \in B$. Then $g(x) = f(x)$ for all $x \in W$. □

We obtain the following result.

Theorem 4.1 *Every topological vector space X is a prehausdorff space, that is, for each $x, y \in X$, $x \neq y$, there exists a preneighbourhood U of x and a preneighbourhood V of y such that $U \cap V = \emptyset$.*

Proof Let $x, y \in X$ and $x \neq y$. If x, y are linearly dependent we choose a linear functional on the span of $\{x, y\}$ such that $f(x) < f(y)$. If x, y are linearly independent we set $f(sx+ty) = t$. By Lemma 4.1 we extend f to a linear functional g with $g(x) < g(y)$. Choose $c \in (g(x), g(y))$ and define $U = g^{-1}((-\infty, c))$, and $V = g^{-1}((c, \infty))$. Then, using Theorem 2.3, U, V are preopen. Also U and V are disjoint and $x \in U$, $y \in V$. □

We now determine totally prebounded subsets in $\mathbb{R}$. The result may not be surprising but the proof requires some care.

Lemma 4.2 *A subset of $\mathbb{R}$ is totally prebounded if and only if it is finite.*

proof It is clear that a finite set is totally prebounded. Let E be a countable (finite or infinite) subset of $\mathbb{R}$ which is totally prebounded. Let $A := \{x - y : x, y \in E\}$. The set A is countable. We define a sequence $\{u_n\}$ of real numbers inductively as follows. We set $u_1 = 0$. Then we choose $u_2 \in (-1, 0)$ such that $u_2 - u_1 \notin A$. Then we choose $u_3 \in (0, 1)$ such that $u_3 - u_i \notin A$ for $i = 1, 2$. Then we choose $u_4 \in (-1, -\frac{1}{2})$ such that $u_4 - u_i \notin A$ for $i = 1, 2, 3$. Continuing in this way we construct a set $U = \{u_n : n \in \mathbb{N}\} \subset (-1, 1)$ such that every interval of the form $(m2^{-k}, (m + 1)2^{-k})$ with $-2^k \leqslant m < 2^k$, $k \in \mathbb{N}$, contains at least one element of U, and such that $0 \in U$ and $u - v \notin A$ for all $u, v \in U$, $u \neq v$. Then $\mathrm{cl}(U) = [-1, 1]$ so U is a preneighborhood of 0. Since E is totally prebounded, there is a finite set F such that $E \subset F + U$. If $z \in F$ and $x, y \in E$ lie in $z + U$ then $x = z + u$, $y = z + v$ with $u, v \in U$. It follows that $u - v = x - y \in A$ and, by construction of U, $u = v$. Therefore, $x = y$ and so each set $z + U$, $z \in F$, contains at most one element of E. Therefore, E is finite. We have shown that every countable set which is totally prebounded is finite. It follows hat every totally prebounded set is finite. □

Combining several of our results we can now identify totally prebounded and precompact subset of any topological vector space.

Theorem 4.2 *Let X be a topological vector space. A subset of X is totally prebounded if and only if it is finite. Similarly, a subset of X is precompact if and only it is finite.*

Proof Every finite set is totally prebounded. Conversely, suppose that E is a totally

prebounded subset of X. Let f be a linear functional on X. It follows easily from Theorem 2.3 that $f(E)$ is a totally prebounded subset of $\mathbb{R}$. By Lemma 4.2, $f(E)$ is finite. It follows that E is finite as we see as follows. Suppose that E contains a sequence $\{x_n\}_{n=1}^{\infty}$ which is linearly independent. Then, using Lemma 4.1, we can construct a linear functional f on X such that $f(x_n) \neq f(x_m)$ if $n \neq m$. This is a contradiction so E must lie in a finite dimensional subspace Y of X. We choose a basis $y_1, \ldots, y_k$ in Y, and represent each $x \in E$ in this basis

$$x = f_1(x)y_1 + \cdots + f_k(x)y_k.$$

Every f_j is a linear functional on Y so $f_j(E)$ is a finite set for each $j = 1, 2, \ldots, k$. It follows that E is finite.

Clearly, every finite set is precompact. Conversely, by Lemma 3.1, a precompact subset of X is totally prebounded, so it is finite. □

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Path Double Covering Number of Product Graphs

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Abstract: A path partition or a path cover of a graph G is a collection P of paths in G such that every edge of G is in exactly one path in P. Various types of path covers such as Smarandache path k-cover, simple path covers have been studied by several authors by imposing conditions on the paths in the path covers. In this paper, We study the minimum number of paths which cover a graph such that each edge of the graph occurs exactly twice in two(distinct) paths.

Key Words: Path double cover; path double cover number; product graphs; Smarandache path k-cover.

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§1. Introduction

Let G be a simple graph and ψ be a collection of paths covering all the edges of G exactly twice. Then ψ is called a path double cover of G. The notion of path double cover was first introduced by J.A. Bondy in [4]. In the paper, he posed the following conjecture:

Every simple graph has a path double cover ψ such that each vertex of G occurs exactly twice as an end of a path of ψ.

The above conjecture was proved by Hao Li in [5] and the conjecture becomes a theorem now. The theorem implies that every simple graph of order p can be path double covered by at most p paths. Obviously, the reason We need p paths in a perfect path double cover is due to the requirement that every vertex must be an end vertex of a path exactly twice. If We drop this requirement, the number of paths need is less than p in general. In this paper, We shall investigate the following number:

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$$\gamma_2(G) = \min \{ |\psi| : \psi \text{ is a path double cover of } G \}$$

For convenience, We call $\gamma_2(G)$, the path double cover number of G throughout this paper.

We need the following definitions for our discussion. For two graphs G and H their cartesian product $G \times H$ has vertex set $V(G) \times V(H)$ in which (g_1, h_1) is adjacent to (g_2, h_2) if $g_1 = g_2$ and $h_1h_2 \in E(H)$ or $h_1 = h_2$ and $g_1g_2 \in E(G)$. For the graphs G and H their wreath product $G * H$ has vertex set $V(G) \times V(H)$ in which (g_1, h_1) is joined to (g_2, h_2) whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. Similarly $G \circ H$, the weak product of graphs G and H has vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. We define the vertex set of the weak product of graphs as in [6] and hence $V(G \circ H) = V_1 \cup V_2 \cup \ldots \cup V_n$ where $V_i = \{u_1^i, u_2^i, \ldots, u_m^i\}, 1 \leq i \leq n$, u_j^i stands for (v_i, u_j) and $V(H) = \{u_1, u_2, \ldots, u_m\}$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Clearly, for each edge $v_iv_j \in E(G)$ the subgraph of $G \circ K_m$ induced by $V_i \cup V_j$ is $K_{V_i,V_j} \setminus \alpha_1(V_i, V_j)$, where $\alpha_k(V_i, V_j)$ is a 1-factor given by $\alpha_k(V_i, V_j) = (u_1^iu_k^j, u_2^iu_{k+1}^j, u_3^iu_{k+2}^j, \ldots, u_m^iu_{k-1}^j), 1 \leq k \leq m$.

For our future reference we state some known results.

Theorem 1.1([3]) *If both G_1 and G_2 have Hamilton cycle decomposition and at least one of G_1 and G_2 is of odd order then $G_1 \circ G_2$ has a Hamilton cycle decomposition.*

Theorem 1.2([5]) *Let $m \geq 3$ and $n \geq 3$. The graph $C_m \times C_n$ can be decomposed into two Hamilton cycles if and only if at least one of the numbers m, n is odd.*

Theorem 1.3([7]) *Let $m \geq 3$and $n \geq 2$. If mis even, then $C_m \circ P_n$ consists of two connected components which are isomorphic to each other.*

Theorem 1.4([7]) *If $m = 4i + 2$, $i \geq 1$, and $n \geq 2$, then each connected component of the graph $C_m \circ P_n$ is isomorphic to $C_{m/2} \circ P_n$.*

A more general definition on graph covering using paths is given as follows.

Definition 1.5([2]) *For any integer $k \geq 1$, a Smarandache path k-cover of a graph G is a collection ψ of paths in G such that each edge of G is in at least one path of ψ and two paths of ψ have at most k vertices in common. Thus if k = 1 and every edge of G is in exactly one path in ψ, then a Smarandache path k-cover of G is a simple path cover of G.*

§2. Main results

Lemma 2.1 *Let G be a graph with n pendent vertices. Then $\gamma_2(G) \geq n$.*

Proof Every pendent vertex is an end vertex of two different paths of a path double cover of G. Since there are n pendent vertices, We have $\gamma_2(G) \geq n$. $\square$

Lemma 2.2 *If G is a graph with $\delta(G) \geq 2$, then $\gamma_2(G) \geq max(\delta(G) + 1, \Delta(G))$.*

Proof One can observe that the total degree of each vertex v of G in a path double cover is $2deg(v)$. If v is an external vertex of a path in a path double cover ψ of G then v is external in at least two different paths of ψ. So We have

$$\begin{aligned} |\psi| &\geq \{(2deg(v)-2)/2\}+2 \\ &= deg(v)-1+2 = deg(v)+1 \geq \delta(G)+1 \end{aligned}$$

This is true for every path double cover of G. Hence $\gamma_2(G) \geq \delta(G)+1$. Let u be a vertex of degree Δ in G. We always have $|\psi| \geq 2deg(u)/2 = \Delta$. Hence $\gamma_2(G) \geq max(\delta(G)+1, \Delta(G))$.□

Corollary 2.3 *If G is a k-regular graph, then $\gamma_2(G) \geq k+1$ and for all other graphs $\gamma_2(G) \geq \Delta$.*

Proof We know that $\Delta(G) \geq \delta(G)$ and for a regular graph $\delta(G) = \Delta(G)$. Hence the result follows. □

Proposition 2.4 *Letm ≥ 3.*

$$\gamma_2(C_m \circ K_2) = \begin{cases} 3 & \text{if m is odd;} \\ 6 & \text{if m is even.} \end{cases}$$

Proof The proof is divided into two cases.

Case (i) m is odd.

Since $C_m \circ K_2$ is a 2-regular graph, We have $\gamma_2(C_m \circ K_2) \geq 3$ by Corollary 2.3. Now We prove the other part.

The graph $C_m \circ K_2$ is a hamilton cycle C of length $2m$. (ie) $C =< u_1u_2u_3 \ldots u_{2m}u_1 >$. Then we take

$P_1 =< u_2u_3 \ldots u_{2m-1}u_{2m} >$;

$P_2 =< u_3u_4 \ldots u_{2m-1}u_{2m}u_1 >$;

$P_3 =< u_1u_2u_3 >$;

The above three paths form the path double cover for $C_m \circ K_2$ and $\gamma_2(C_m \circ K_2) \leq 3$. Hence $\gamma_2(C_m \circ K_2) = 3$.

Case (ii) m is even.

The graph $C_m \circ K_2$ can be factorized into two isomorphic components of cycle of even length m. Also since any cycle has minimum path double cover number as 3 (by using Case(i)). We have $\gamma_2(C_m \circ K_2) = 6$. □

Proposition 2.5 *Let $m, n \geq 3$. $\gamma_2(C_m \circ C_n) = 5$ if at least one of the numbers m and n is odd.*

Proof Since $C_m \circ C_n$ is a 4- regular graph, We have $\gamma_2(C_m \circ C_n) \geq 5$ by Corollary 2.3. Since at least one of the numbers m and n is odd, $C_m \circ C_n$ can be decomposed into two hamilton

cycles C_1 and C_2 by Theorem 1.1. Let $u \in V(C_m \circ C_n)$. Since $deg(u) = 4$, there exist four vertices v_1, v_2, v_3 and v_4 adjacent with u and exactly two of them together with u are on C_1 and the other two together with u are on C_2. Without loss of generality assume that $< v_1 u v_2 >$ and $< v_3 u v_4 >$ lie on the cycles C_1 and C_2 respectively.

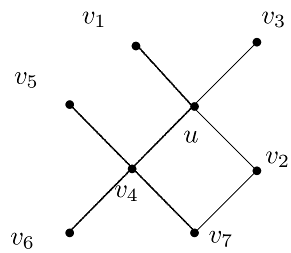


Fig. 1

Since $deg(v_4) = 4$, there are vertices v_5, v_6, v_7 together with u are adjacent with v_4 as in Fig.1. Now assume that $< v_5 v_4 v_7 >$ and $< u v_4 v_6 >$ lie on C_1 and C_2 respectively. Let $C_i^{(1)}, C_i^{(2)}$ be the two copies of C_i $(i = 1, 2)$. If $v_2 v_7$ is in C_1 then $\{(C_1^{(1)} - (v_2 v_7)), (C_1^{(2)} - (v_4 v_7)), (C_2^{(1)} - (u v_3)), (C_2^{(2)} - (u v_4)), (v_3 u v_4 v_7 v_2)\}$ is a path double cover for $C_m \circ C_n$. Otherwise $v_2 v_7$ is in C_2 and $\{(C_1^{(1)} - (v_1 u)), (C_1^{(2)} - (v_4 v_7)), (C_2^{(1)} - (u v_4)), (C_2^{(2)} - (v_2 v_7)), (v_1 u v_4 v_7 v_2)\}$ is a path double cover for $C_m \circ C_n$. Hence $\gamma_2(C_m \circ C_n) = 5$. For the remaining possibilities it is verified that $\gamma_2(C_m \circ C_n) = 5$ in a similar manner. □

Proposition 2.6 *Let $m, n \geq 3$.*

$$\gamma_2(P_m \circ C_n) = \begin{cases} 4 & \text{if n } \equiv 1 \text{ or } 3 (\text{mod } 4) \\ 8 & \text{if n } \equiv 0 \text{ or } 2 (\text{mod } 4) \end{cases}$$

Proof Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ and let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$.
Let $V_i = \{u_1^i, u_2^i, \ldots, u_n^i\}, 1 \leq i \leq m$ be the set of n vertices of $P_m \circ C_n$ that corresponds to the vertex v_i of P_m.

Case (i) When $n \equiv 1$ or $3(\text{mod } 4)$, n must be odd.By Lemma 2.2, $\gamma_2(P_m \circ C_n) \geq 4$. Now we prove the other part. Take

$P_1 =< u_1^1 u_2^2 u_3^1 u_4^2 \ldots u_{n-1}^2 u_n^1 u_1^2 u_2^3 u_3^2 \ldots u_{n-1}^m u_n^{m-1} u_1^m >= P_3$.

$P_2 =< u_1^1 u_n^2 u_{n-1}^1 u_{n-2}^2 \ldots u_2^1 u_1^2 u_n^3 u_{n-1}^2 \ldots u_2^{m-1} u_1^m >= P_4$.

Clearly $\{P_1, P_2, P_3, P_4\}$ is a path double cover for $P_m \circ C_n$ and $\gamma_2(P_m \circ C_n) \leq 4$. Hence $\gamma_2(P_m \circ C_n) = 4$.

Case (ii) When $n \equiv 2(\text{mod } 4)$, $n = 4i + 2$ where $i \geq 1$,then by Theorems 1.3 and 1.4, $P_m \circ C_n$ can be decomposed into two connected components and each of which is isomorphic to $P_m \circ C_{n/2}$. Now since $n/2$ is odd, using Case(i) We can have $\gamma_2(P_m \circ C_{n/2}) = 4$ and hence $\gamma_2(P_m \circ C_n) = 4$.

Case (iii) When $n \equiv 0(mod4)$, $n = 4i$ where $i \geq 1$.

Subcase (1) m is even.

$$P_1 =< u_1^1u_2^2u_3^1u_4^2 \ldots u_{n-1}^1u_n^2u_{n-1}^3u_{n-2}^4u_{n-3}^3u_{n-4}^4 \ldots u_2^4u_1^3u_1^4u_n^4u_{n-1}^5 \\ u_{n-2}^6u_{n-3}^5 \ldots u_1^{m-1}u_n^m >= P_5;$$

$$P_2 =< u_1^1u_n^2u_1^3u_2^2u_3^3u_4^2u_3^5 \ldots u_{n-2}^2u_{n-1}^3u_n^4u_1^5u_2^4u_3^5 \ldots u_{n-2}^4u_{n-1}^5 \\ u_n^6u_1^7 \ldots u_{n-2}^{m-2}u_{n-1}^{m-1}u_n^m >= P_6;$$

$$P_3 =< u_n^1u_{n-1}^2u_{n-2}^1u_{n-3}^2 \ldots u_2^1u_1^2u_2^3u_3^4u_4^3u_5^4 \ldots u_{n-1}^4u_n^3u_1^4u_2^5 \\ u_3^6u_4^5 \ldots u_{n-1}^6u_n^5u_1^6 \ldots u_{n-1}^mu_n^{m-1}u_1^m >= P_7;$$

$$P_4 =< u_n^1u_1^2u_n^3u_{n-1}^2u_{n-2}^3u_{n-3}^2 \ldots u_3^2u_2^3u_1^4u_n^5u_{n-1}^4u_{n-2}^5 \ldots u_4^{m-1}u_3^{m-2}u_2^{m-1}u_1^m >= P_8.$$

Subcase (2) m is odd.

$$P_1 =< u_1^1u_2^2u_3^1u_4^2 \ldots u_{n-1}^1u_n^2u_{n-1}^3u_{n-2}^4u_{n-3}^3u_{n-4}^4 \ldots u_2^4u_1^3u_n^4u_{n-1}^5 \\ u_{n-2}^6u_{n-3}^5 \ldots u_n^{m-1}u_{n-1}^m >= P_5;$$

$$P_2 =< u_1^1u_n^2u_1^3u_2^2u_3^3u_4^2u_3^5 \ldots u_{n-2}^2u_{n-1}^3u_n^4u_1^5u_2^4u_3^5 \ldots u_{n-2}^4u_{n-1}^5 \\ u_n^6u_1^7 \ldots u_{n-3}^mu_{n-2}^{m-1}u_{n-1}^m >= P_6;$$

$$P_3 =< u_n^1u_{n-1}^2u_{n-2}^1u_{n-3}^2 \ldots u_2^1u_1^2u_2^3u_3^4u_4^3u_5^4 \ldots u_{n-1}^4u_n^3u_1^4u_2^5 \\ u_3^6u_4^5 \ldots u_{n-1}^6u_n^5u_1^6 \ldots u_1^{m-1}u_2^m >= P_7;$$

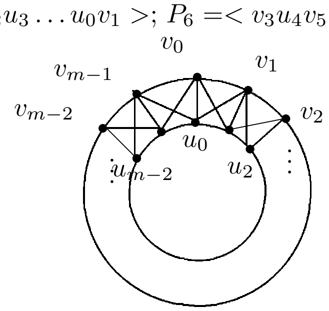
$$P_4 =< u_n^1u_1^2u_n^3u_{n-1}^2u_{n-2}^3u_{n-3}^2 \ldots u_3^2u_2^3u_1^4u_n^5u_{n-1}^4u_{n-2}^5 \ldots u_4^mu_3^{m-1}u_2^m >= P_8.$$

In both Subcases $\{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$ is the minimum path double cover for $P_m \circ C_n$ and hence $\gamma_2(P_m \circ C_n) = 8$ for $n \equiv 0(\text{mod } 4)$. $\Box$

Proposition 2.7 *Let* $m \geq 3$, $\gamma_2(C_m * K_2) = 6$ *if* m *is odd.*

Proof Since the graph is 5-regular, We have $\gamma_2(C_m * K_2) \geq 6$ by Corollary 2.3. Now We prove the converse part. Given m is odd, then take

$P_1 =< u_0u_1v_1v_2u_2u_3v_3 \ldots u_{m-2}v_{m-2}v_{m-1} >$; $P_2 =< u_0v_0v_1u_1u_2v_2v_3u_3 \ldots v_{m-2}u_{m-2}u_{m-1} v_{m-1} >$; $P_3 =< u_1u_2u_3 \ldots u_{m-1}u_0v_0v_{m-1}v_{m-2} \ldots v_2v_1 >$; $P_4 =< u_1u_0u_{m-1}v_{m-1}v_0v_1u_2v_3 >$; $P_5 =< u_3v_3u_4v_5 \ldots u_{m-1}v_0u_1v_2u_3 \ldots u_0v_1 >$; $P_6 =< v_3u_4v_5 \ldots u_{m-1}v_0u_1v_2u_3 \ldots u_0v_1u_2 >$.



$C_m * K_2$

Fig.2

$\{P_1, P_2, P_3, P_4, P_5, P_6\}$ is a path double cover for $C_m * K_2$ and $\gamma_2(C_m * K_2) \leq 6$. Hence $\gamma_2(C_m * K_2) = 6$. $\Box$

Proposition 2.8 *Let $m \geq 3$. $\gamma_2(C_m * \bar{K}_2) = 5$ if m is odd.*

Proof By Corollary 2.3, $\gamma_2(C_m * \bar{K}_2) \geq 5$. Given m is odd and take

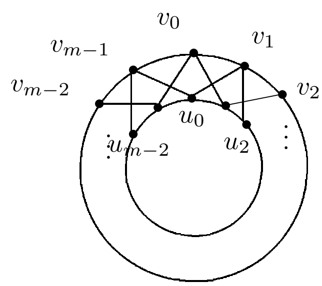
$P_1 =< u_0u_1v_2u_3v_4 \ldots u_{m-2}v_{m-1}v_0u_{m-1}v_{m-2}u_{m-3} \ldots v_3u_2v_1 >$;

$P_2 =< u_0u_1u_2 \ldots u_{m-1}v_0v_1v_2 \ldots v_{m-1} >$;

$P_3 =< u_{m-1}u_0v_{m-1}u_{m-2}v_{m-3} \ldots u_1v_0v_1u_3v_3 \ldots u_{m-3}v_{m-2} >$;

$P_4 =< v_1u_0u_{m-1}u_{m-2}u_{m-3}u_2u_1v_0v_{m-1}v_{m-2} >$;

$P_5 =< v_{m-1}u_0v_1v_2v_3 \ldots v_{m-2}u_{m-1} >$;



$C_m * \bar{K}_2$

Fig.3

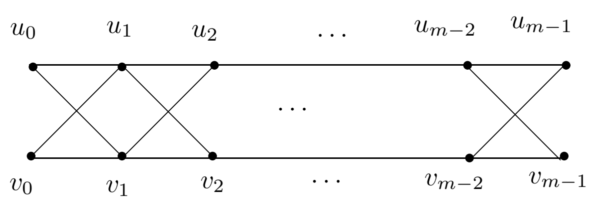
$\{P_1, P_2, P_3, P_4, P_5\}$ is a path double cover for $C_m * \bar{K}_2$ and $\gamma_2(C_m * \bar{K}_2) \leq 5$. Hence $\gamma_2(C_m * \bar{K}_2) = 5$. □

Proposition 2.9 $\gamma_2(P_m * = K_2) = 4$ *for* $m \geq 3$.

Proof By Lemma 2.2, $\gamma_2(P_m * \bar{K}_2) \geq 4$. If m is odd then take

$P_1 =< u_0v_1u_2v_3u_4 \ldots v_{m-2}u_{m-1}u_{m-2}v_{m-3} \ldots v_2u_1v_0 >= P_3$;

$P_2 =< u_0u_1u_2 \ldots u_{m-2}v_{m-1}v_{m-2} \ldots v_1v_0 >= P_4$.



$P_m * \bar{K}_2$

Fig.4

If m is even then take

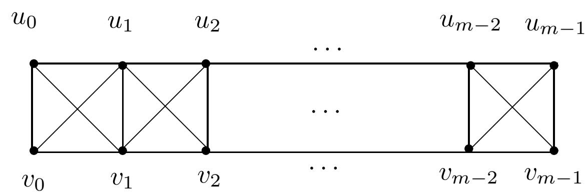
$P_1 =< u_0v_1u_2v_3u_4 \ldots u_{m-2}v_{m-1}v_{m-2}u_{m-3} \ldots v_2u_1v_0 >= P_3$;

$P_2 =< u_0u_1u_2 \ldots u_{m-1}v_{m-2}v_{m-3} \ldots v_1v_0 >= P_4$.

Clearly $\{P_1, P_2, P_3, P_4\}$ is a path double cover for $P_m * \bar{K}_2$ and $\gamma_2(P_m * \bar{K}_2) \leq 4$. Hence $\gamma_2(P_m * \bar{K}_2) = 4$. □

Proposition 2.10 *$\gamma_2(P_m * K_2) = 5$ for $m \geq 3$.*

Proof By Lemma 2.2, $\gamma_2(P_m * K_2) \geq 5$.



$P_m * K_2$

Fig.5

If m is even then take

$P_1 =< u_0v_1u_1v_2u_2v_3 \ldots u_{m-2}v_{m-1} >$;

$P_2 =< u_0v_0u_1v_1u_2v_2 \ldots v_{m-2}u_{m-1} >$;

$P_3 =< u_0v_1u_2v_3u_4 \ldots v_{m-3}u_{m-2}v_{m-1}u_{m-1}v_{m-2}u_{m-3} \ldots u_2v_1u_0 >$;

$P_4 =< u_0u_1u_2u_3 \ldots u_{m-1}v_{m-1}v_{m-2} \ldots v_1v_0 >$;

$P_5 =< u_{m-1}u_{m-2}u_{m-3} \ldots u_1u_0v_0v_1 \ldots v_{m-2}v_{m-1} >$.

If m is odd then take

$P_1 =< u_0v_1u_1v_2u_2 \ldots u_{m-2}v_{m-1} >$;

$P_2 =< u_0v_0u_1v_1u_2v_2 \ldots v_{m-2}u_{m-1} >$;

$P_3 =< u_0v_1u_2v_3u_4v_4 \ldots v_{m-2}u_{m-1}v_{m-1}u_{m-2} \ldots v_2u_1v_0 >$;

$P_4 =< u_0u_1u_2 \ldots u_{m-1}v_{m-1}v_{m-2} \ldots v_1v_0 >$;

$P_5 =< u_{m-1}u_{m-2} \ldots u_1u_0v_0v_1 \ldots v_{m-1} >$.

$\{P_1, P_2, P_3, P_4, P_5\}$ is a path double cover for $P_m * K_2$ and $\gamma_2(P_m * K_2) \leq 5$. Hence $\gamma_2(P_m * K_2) = 5$. □

Proposition 2.11 *Let $m \geq 3$. $\gamma_2(C_m \times P_3) = 5$ if m is odd.*

Proof Let $V(C_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(P_3) = \{u_1, u_2, u_3\}$. Let $V_i = \{u_1^i, u_2^i, u_3^i\}, 1 \leq i \leq m$ be the set of 3 vertices of $C_m \times P_3$ that corresponds to a vertex v_i of C_m. Now we construct the paths for $C_m \times P_3$.

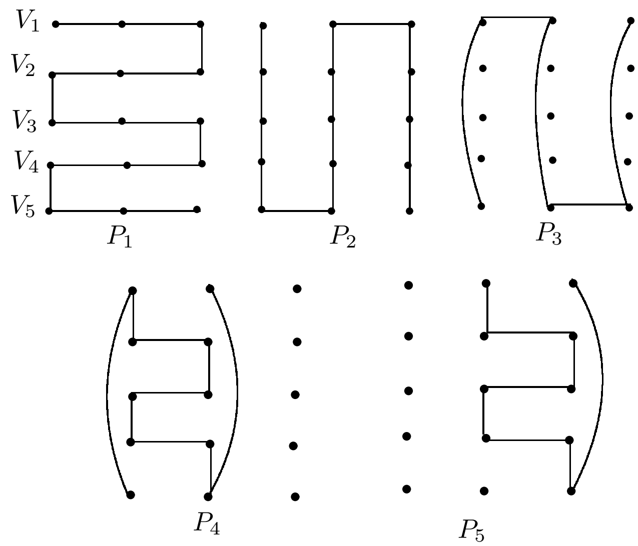
$P_1 =< u_1^1u_2^1u_3^1u_3^2u_2^2u_1^2u_1^3u_2^3u_3^3 \ldots u_1^mu_2^mu_3^m >$;

$P_2 =< u_1^1u_1^2u_1^3 \ldots u_1^mu_2^mu_2^{m-1} \ldots u_2^2u_2^1u_3^1u_3^2u_3^m >$;

$P_3 =< u_1^mu_1^1u_2^1u_2^mu_3^mu_3^1 >$;

$P_4 =< u_1^mu_1^1u_1^2u_2^2u_2^3u_1^3u_1^4u_2^4 \ldots u_2^{m-1}u_2^mu_2^1 >$;

$P_5 =< u_2^1u_2^2u_3^2u_3^3u_2^3u_2^4 \ldots u_3^mu_3^1 >$.



$C_5 \times P_3$

Fig.6

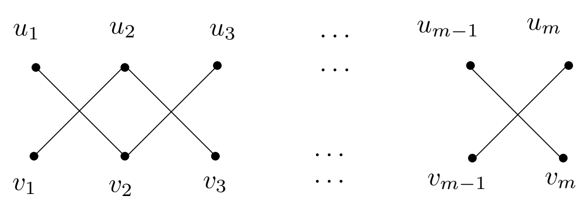
Clearly $\{P_1, P_2, P_3, P_4, P_5\}$ is a path double cover for $C_m \times P_3$ (See Fig.6) and $\gamma_2(C_m \times P_3) \leq 5$. Hence $\gamma_2(C_m \times P_3) = 5$. □

Proposition 2.12 *$\gamma_2(P_m \circ K_2) = 4$ for $m \geq 2$.*

Proof By Theorem 2.1, $\gamma_2(P_m \circ K_2) \geq 4$. If m is even then take

$P_1 =< u_1v_2u_3v_4u_5 \dots v_{m-2}u_{m-1}v_m >= P_3$;

$P_2 =< v_1u_2v_3u_4v_5 \dots u_{m-2}v_{m-1}u_m >= P_4$.



$P_m \circ K_2$

Fig.7

If m is odd then take

$P_1 =< u_1v_2u_3v_4u_5 \dots u_{m-2}v_{m-1}u_m >= P_3$;

$P_2 =< v_1u_2v_3u_4v_5 \dots v_{m-2}u_{m-1}v_m >= P_4$.

$\{P_1, P_2, P_3, P_4\}$ is a path double cover for $P_m \circ K_2$ and $\gamma_2(P_m \circ K_2) \leq 4$. Hence $\gamma_2(P_m \circ K_2) = 4$. □

Proposition 2.13 *For the complete bipartite graph $K_{m,n}$, $\gamma_2(K_{m,n}) = \max\{m, n\}$.*

Proof Let (A, B) be the bipartition of $K_{m,n}$ where $A = \{u_0, u_1, \ldots, u_{m-1}\}$, $B = \{v_0, v_1, \ldots, v_{n-1}\}$.

Case (i) $m \leq n$.

By Corollary 2.3, $\gamma_2(K_{m,n}) \geq n$. Let $P_i =< v_i u_1 v_{i+1} u_2 \ldots u_{m-1} v_{i+m-1} u_0 v_{i+m} >$, where the indices i are taken modulo n. $\psi = \{P_i : 0 \leq i \leq n-1\}$ is clearly a path double cover for $K_{m,n}$ with n paths. Hence $\gamma_2(K_{m,n}) = n$.

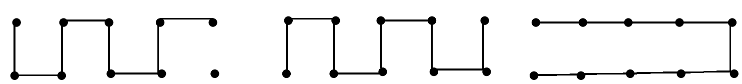
Case (ii) $m > n$.

By Corollary 2.3, $\gamma_2(K_{m,n}) \geq m$.Let $P_i =< u_i v_1 u_{i+1} v_2 \ldots v_{n-1} u_{i+n-1} v_0 u_{i+n} >$, where the indices i are taken modulo m. $\psi = \{P_i : 0 \leq i \leq m-1\}$ is clearly a path double cover for $K_{m,n}$ with m paths. Hence $\gamma_2(K_{m,n}) = m$. This completes the proof. $\square$

Proposition 2.14 *Let* $m, n \geq 2$.

$$\gamma_2(P_m \times P_n) = \begin{cases} 3 & \text{if m=2 or n=2;} \\ 4 & \text{otherwise.} \end{cases}$$

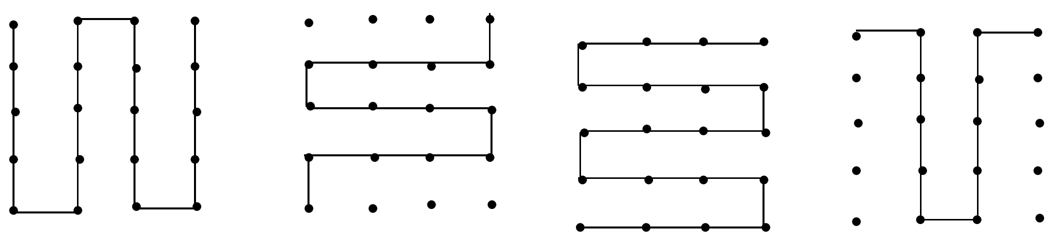
Proof By Lemma 2.2, $\gamma_2(P_m \times P_n) \geq 3$ if $m = 2$ or $n = 2$ and $\gamma_2(P_m \times P_n) \geq 4$ if $m, n \geq 3$. The reverse inclusion follows from Fig.8 and Fig.9.



P_1 P_2 P_3

$P_2 \times P_5$

Fig.8



$P_5 \times P_4$

Fig.9

$\square$

Proposition 2.15 *Let* $m \geq 3$, $n \geq 3$. $\gamma_2(C_m \times C_n) = 5$ *if at least one of the numbers* m *and* n *is odd.*

Proof Since $C_m \times C_n$ is a 4-regular graph, We have $\gamma_2(C_m \times C_n) \geq 5$ by corollary 2.3. Since at least one of the numbers m and n is odd, $C_m \times C_n$ can be decomposed into two

hamilton cycles C_1 and C_2 by Theorem 1.2. Let $v \in V(C_m \times C_n)$. Since $deg(v) = 4$, there exist four vertices u_1, u_2, u_3 and u_4 adjacent with v and exactly two of them together with v are on C_1 and the other two together with v are on C_2. Without loss of generality assume that $< u_1vu_2 >$ and $< u_3vu_4 >$ lie on C_1 and C_2 respectively. Since $deg(u_4) = 4$, there are vertices u_5, u_6, u_7 together with v are adjacent with u_4 as in Fig.10. As before assume that $(u_5u_4u_6)$ and (vu_4u_7) lie on C_1 and C_2 respectively. Let $C_i^{(1)}, C_i^{(2)}$ be the two copies of $C_i (i = 1, 2)$. If u_2u_6 is in C_1 then $\{(C_1^{(1)} - (u_2u_6)), (C_1^{(2)} - (u_4u_6)), (C_2^{(1)} - (vu_3)), (C_2^{(2)} - (vu_4)), (u_3vu_4u_6u_2)\}$ is a path double cover for $C_m \times C_n$. Otherwise u_2u_6 is in C_2 and $\{(C_1^{(1)} - (u_1v)), (C_1^{(2)} - (u_4u_6)), (C_2^{(1)} - (vu_4)), (C_2^(2) - (u_2u_6)), (u_1vu_4u_6u_2)\}$ is a path double cover for $C_m \times C_n$. Hence $\gamma_2(C_m \times C_n) = 5$. For the remaining possibilities it is verified that $\gamma_2(C_m \times C_n) = 5$ in a similar manner.

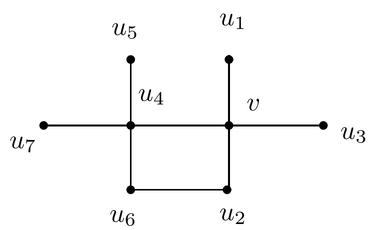


Fig.10

□

Proposition 2.16 $\gamma_2(C_m \times K_2) = 4$ *for* $m \geq 3$.

Proof By Corollary 2.3, $\gamma_2(C_m \times K_2) \geq 4$. Now We prove the other part. If m is odd then take

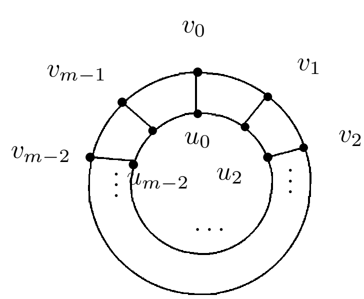
$P_1 =< u_0u_1v_1v_2u_2u_3v_3 \ldots u_{m-2}v_{m-2}v_{m-1} >$;

$P_2 =< u_0v_0v_1u_1u_2v_2v_3u_3 \ldots v_{m-2}u_{m-2}u_{m-1}v_{m-1} >$.

If m is even then take

$P_1 =< u_0u_1v_1v_2u_2u_3v_3 \ldots v_{m-2}u_{m-2}u_{m-1} >$;

$P_2 =< u_0v_0v_1u_1u_2v_2v_3u_3 \ldots u_{m-2}v_{m-2}v_{m-1}u_{m-1} >$.



$C_m \times K_2$
Fig.11

Let $P_3 =< u_1u_2 \ldots u_{m-1}u_0v_0v_{m-1}v_{m-2} \ldots v_2v_1 >$ and

$P_4 =< u_1u_0u_{m-1}v_{m-1}v_0v_1 >$.

$\{P_1, P_2, P_3, P_4\}$ is a path double cover for $C_m \times K_2$ and $\gamma_2(C_m \times K_2) \leq 4$. Hence $\gamma_2(C_m \times K_2) = 4$. $\Box$

Proposition 2.17 *$\gamma_2(K_{2n+1} \circ K_2) = 2n + 1$, for $n \geq 1$.*

Proof Let $V(K_{2n+1}) = \{v_0, v_1, v_2, \ldots, v_{2n}\}$ and let $V(K_2) = \{u_0, u_1\}$. Let $V_i = \{u_0^i, u_1^i\}$, $0 \leq i \leq 2n$ be the set of $2n + 1$ vertices of $K_{2n+1} \circ K_2$ that corresponds to a vertex v_i of K_{2n+1}. Define for $1 \leq i \leq 2n$, $H_i =< v_0v_iv_{i+1}v_{i-1}v_{i+2}v_{i-2} \ldots v_{n+i-1}v_{n+i+1}v_{n+i}v_0 >$ (H_i is nothing but the Walecki's Hamilton cycle decomposition[1]) where the indices are taken modulo $2n$ except 0. Clearly $\{H_1, H_2, \ldots, H_{2n}\}$ is a cycle double cover for K_{2n+1}. Then

$$\begin{aligned} K_{2n+1} \circ K_2 &= (H_1 \oplus H_2 \oplus \ldots \oplus H_{2n}) \circ K_2 \\ &= H_1 \circ K_2 \oplus H_2 \circ K_2 \oplus \ldots \oplus H_{2n} \circ K_2 \end{aligned}$$

where $H_i \circ K_2 (1 \leq i \leq 2n)$ is a cycle double cover for $K_{2n+1} \circ K_2$. Now remove an edge e_i from $H_i \circ K_2 (1 \leq i \leq 2n)$ so that $< e_1e_2e_3 \ldots e_{2n} >$ form a path (See Example 2.18). Hence $\gamma_2(K_{2n+1} \circ K_2) = 2n + 1$. Since $\delta(K_{2n+1} \circ K_2) = 2n$, We have $\gamma_2(K_{2n+1} \circ K_2) \geq 2n + 1$, by Lemma 2.2. Hence $\gamma_2(K_{2n+1} \circ K_2) = 2n + 1$. $\Box$

The following example illustrates the above theorem.

Example 2.18 Consider $K_5 \circ K_2$, $H_i =< v_0v_iv_{i+1}v_{i-1}v_{i+2}v_0 >, 1 \leq i \leq 4$. The cycle double covers for $K_{2n+1} \circ K_2$ are

$H_1 \circ K_2 =< u_1^0u_2^1u_1^2u_2^4u_1^3u_2^0u_1^1u_2^2u_1^4u_2^3u_1^0 >$;

$H_2 \circ K_2 =< u_1^0u_2^2u_1^3u_2^1u_1^4u_2^0u_1^2u_2^3u_1^1u_2^4u_1^0 >$;

$H_3 \circ K_2 =< u_1^0u_2^3u_1^4u_2^2u_1^1u_2^0u_1^3u_2^4u_1^2u_2^1u_1^0 >$ and

$H_4 \circ K_2 =< u_1^0u_2^4u_1^1u_2^3u_1^2u_2^0u_1^4u_2^1u_1^3u_2^2u_1^0 >$.

Now remove the edges $(u_1^0u_2^1), (u_2^1u_1^4), (u_1^4u_2^3)$ and $(u_2^3u_1^1)$ from$H_1 \circ K_2$, $H_2 \circ K_2, H_3 \circ K_2$ and $H_4 \circ K_2$ respectively so that $< u_1^0u_2^1u_1^4u_2^3u_1^1 >$ form a path. Hence $\gamma_2(K_5 \circ K_2) = 5$.

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Some Remarks on Fuzzy N-Normed Spaces

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Abstract: It is shown that every fuzzy n-normed space naturally induces a locally convex topology, and that every finite dimensional fuzzy n-normed space is complete.

Key Words: Fuzzy n-normed spaces, n-seminorm, Smarandache space.

AMS(2000): 34A10, 34C10.

§1. Introduction

A Smarandache space is such a space that a straight line passing through a point p may turn an angle $\theta_p \geq 0$. If $\theta_p > 0$, then p is called a non-Euclidean. Otherwise, we call it an Euclidean point. In this paper, normed spaces are considered to be Euclidean, i.e., every point is Euclidean. In [7], S. Gähler introduced n-norms on a linear space. A detailed theory of n-normed linear space can be found in [8], [10], [12]-[13]. In [8], H. Gunawan and M. Mashadi gave a simple way to derive an $(n-1)$- norm from the n-norm in such a way that the convergence and completeness in the n-norm is related to those in the derived $(n-1)$-norm. A detailed theory of fuzzy normed linear space can be found in [1], [3]-[6], [9], [11] and [15]. In [14], A. Narayanan and S. Vijayabalaji have extend n-normed linear space to fuzzy n-normed linear space. In section 2, we quote some basic definitions, and we show that a fuzzy n-norm is closely related to an ascending system of n-seminorms. In Section 3, we introduce a locally convex topology in a fuzzy n-normed space, and in Section 4 we consider finite dimensional fuzzy n-normed linear spaces.

§2. Fuzzy n-norms and ascending families of n-seminorms

Let n be a positive integer, and let X be a real vector space of dimension at least n. We recall the definitions of an n-seminorm and a fuzzy n-norm [14].

Definition 2.1 *A function $(x_1, x_2, \ldots, x_n) \mapsto \|x_1, \ldots, x_n\|$ from X^n to $[0, \infty)$ is called an n-seminorm on X if it has the following four properties:*

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(S1) $\|x_1, x_2, \ldots, x_n\| = 0$ *if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(S2) $\|x_1, x_2, \ldots, x_n\|$ *is invariant under any permutation of* $x_1, x_2, \ldots, x_n$*;*

(S3) $\|x_1, \ldots, x_{n-1}, cx_n\| = |c|\|x_1, , \ldots, x_{n-1}, x_n\|$ *for any real c;*

(S4) $\|x_1, \ldots, x_{n-1}, y+z\| \leqslant \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\|$.

An n-seminorm is called a n-norm if $\|x_1, x_2, \ldots, x_n\| > 0$ *whenever* $x_1, x_2, \ldots, x_n$ *are linearly independent.*

Definition 2.2 *A fuzzy subset N of $X^n \times \mathbb{R}$ is called a fuzzy n-norm on X if and only if:*

(F1) For all $t \leqslant 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$*;*

(F2) For all $t > 0$, $N(x_1, x_2, \ldots, x_n, t) = 1$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(F3) $N(x_1, x_2, \ldots, x_n, t)$ *is invariant under any permutation of* $x_1, x_2, \ldots, x_n$*;*

(F4) For all $t > 0$ *and* $c \in \mathbb{R}$, $c \neq 0$,

$$N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|});$$

(F5) For all $s, t \in \mathbb{R}$,

$$N(x_1, \ldots, x_{n-1}, y+z, s+t) \geq \min\left\{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}z, t)\right\}.$$

(F6) $N(x_1, x_2, \ldots, x_n, t)$ *is a non-decreasing function of* $t \in \mathbb{R}$ *and*

$$\lim_{t\to\infty} N(x_1, x_2, \ldots, x_n, t) = 1.$$

The following two theorems clarify the relationship between definitions 2.1 and 2.2.

Theorem 2.1 *Let N be a fuzzy n-norm on X. As in [14] define for $x_1, x_2, \ldots, x_n \in X$ and $\alpha \in (0,1)$*

$$\|x_1, x_2, \ldots, x_n\|_\alpha := \inf\left\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\right\}. \tag{2.1}$$

Then the following statements hold.

(A1) For every $\alpha \in (0,1)$, $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$ *is an n-seminorm on X;*

(A2) If $0 < \alpha < \beta < 1$ *and* $x_1, \ldots, x_n \in X$ *then*

$$\|x_1, x_2, \ldots, x_n\|_\alpha \leqslant \|x_1, x_2, \ldots, x_n\|_\beta;$$

(A3) If $x_1, x_2, \ldots, x_n \in X$ *are linearly independent then*

$$\lim_{\alpha\to 1-} \|x_1, x_2, \ldots, x_n\|_\alpha = \infty.$$

Proof (A1) and (A2) are shown in Theorem 3.4 in [14]. Let $x_1, x_2, \ldots, x_n \in X$ be linearly independent, and $t > 0$ be given. We set $\beta := N(x_1, x_2, \ldots, x_n, t)$. It follows from (F2) that $\beta \in [0, 1)$. Then (F6) shows that, for $\alpha \in (\beta, 1)$,

$$\|x_1, x_2, \ldots, x_n\|_\alpha \geqslant t.$$

This proves (A3). $\square$

We now prove a converse of Theorem 2.2.

Theorem 2.2 *Suppose we are given a family* $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$, $\alpha \in (0, 1)$, *of* n*-seminorms on* X *with properties (A2) and (A3). We define*

$$N(x_1, x_2, \ldots, x_n, t) := \inf\{\alpha \in (0, 1) : \|x_1, x_2, \ldots, x_n\|_\alpha \geqslant t\}. \tag{2.2}$$

where the infimum of the empty set is understood as 1. *Then* N *is a fuzzy* n*-norm on* X.

Proof (F1) holds because the values of an n-seminorm are nonnegative.

(F2): Let $t > 0$. If $x_1, \ldots, x_n$ are linearly dependent then $N(x_1, \ldots, x_n, t) = 1$ follows from property (S1) of an n-seminorm. If $x_1, \ldots, x_n$ are linearly independent then $N(x_1, \ldots, x_n, t) < 1$ follows from (A3).

(F3) is a consequence of property (S2) of an n-seminorm.

(F4) is a consequence of property (S3) of an n-seminorm.

(F5): Let $\alpha \in (0, 1)$ satisfy

$$\alpha < \min\{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}, z, s)\}. \tag{2.3}$$

It follows that $\|x_1, \ldots, x_{n-1}, y\|_\alpha < s$ and $\|x_1, \ldots, x_{n-1}, z\|_\alpha < t$. Then (S4) gives

$$\|x_1, \ldots, x_{n-1}, y + z\|_\alpha < s + t.$$

Using (A2) we find $N(x_1, \ldots, x_{n-1}, y + z, s + t) \geqslant \alpha$ and, since α is arbitrary in (2.3), (F5) follows.

(F6): Definition 2.2 shows that N is non-decreasing in t. Moreover, $\lim_{t\to\infty} N(x_1, \ldots, x_n, t)$ $= 1$ because seminorms have finite values. $\square$

It is easy to see that Theorems 2.1 and 2.2 establish a one-to-one correspondence between fuzzy n-norms with the additional property that the function $t \mapsto N(x_1, \ldots, x_n, t)$ is left-continuous for all $x_1, x_2, \ldots, x_n$ and families of n-seminorms with properties (A2), (A3) and the additional property that $\alpha \mapsto \|x_1, \ldots, x_n\|_\alpha$ is left-continuous for all $x_1, x_2, \ldots, x_n$.

Example 2.3(Example 3.3 in [14]). Let $\|\bullet, \bullet, \ldots, \bullet\|$ be a n-norm on X. Then define $N(x_1, x_2,$ $\ldots, x_n, t) = 0$ if $t \leqslant 0$ and, for $t > 0$,

$$N(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|}.$$

Then the seminorms (2.1) are given by

$$\|x_1, x_2, \ldots, x_n\|_\alpha = \frac{\alpha}{1-\alpha}\|x_1, x_2, \ldots, x_n\|.$$

§3. Locally convex topology generated by a fuzzy n-norm

In this section (X, N) is a fuzzy n-normed space, that is, X is real vector space and N is fuzzy n-norm on X. We form the family of n-seminorms $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$, $\alpha \in (0,1)$, according to Theorem 2.1. This family generates a family $\mathcal{F}$ of seminorms

$$\|x_1, \ldots, x_{n-1}, \bullet\|_\alpha, \quad \text{where } x_1, \ldots, x_{n-1} \in X \text{ and } \alpha \in (0,1).$$

The family $\mathcal{F}$ generates a locally convex topology on X; see [2, Def.(37.9)], that is, a basis of neighborhoods at the origin is given by

$$\{x \in X : p_i(x) \leqslant \epsilon_i \text{ for } i = 1, 2, \ldots, n\},$$

where $p_i \in \mathcal{F}$ and $\epsilon_i > 0$ for $i = 1, 2 \ldots, n$. We call this the locally convex topology generated by the fuzzy n-norm N.

Theorem 3.1 *The locally convex topology generated by a fuzzy n-norm is Hausdorff.*

Proof Given $x \in X$, $x \neq 0$, choose $x_1, \ldots, x_{n-1} \in X$ such that $x_1, \ldots, x_{n-1}, x$ are linearly independent. By Theorem 2.1(A3) we find $\alpha \in (0,1)$ such that $\|x_1, \ldots, x_{n-1}, x\|_\alpha > 0$. The desired statement follows; see [2,Theorem (37.21)]. □

Some topological notions can be expressed directly in terms of the fuzzy-norm N. For instance, we have the following result on convergence of sequences. We remark that the definition of convergence of sequences in a fuzzy n-normed space as given in [16, Definition 2.2] is meaningless.

Theorem 3.2 *Let $\{x_k\}$ be a sequence in X and $x \in X$. Then $\{x_k\}$ converges to x in the locally convex topology generated by N if and only if*

$$\lim_{k\to\infty} N(a_1, \ldots, a_{n-1}, x_k - x, t) = 1 \tag{3.1}$$

for all $a_1, \ldots, a_{n-1} \in X$ and all $t > 0$.

Proof Suppose that $\{x_k\}$ converges to x in (X, N). Then, for every $\alpha \in (0,1)$ and all $a_1, a_2, \ldots, a_{n-1} \in X$, there is K such that, for all $k \geqslant K$, $\|a_1, a_2, \ldots, a_{n-1}, x_k - x\|_\alpha < \epsilon$. The latter implies

$$N(a_1, a_2, \ldots, a_{n-1}, x_k - x, \epsilon) \geqslant \alpha.$$

Since $\alpha \in (0,1)$ and $\epsilon > 0$ are arbitrary we see that (3.1) holds. The converse is shown in a similar way. □

In a similar way we obtain the following theorem.

Theorem 3.3 *Let $\{x_k\}$ be a sequence in X. Then $\{x_k\}$ is a Cauchy sequence in the locally convex topology generated by N if and only if*

$$\lim_{k,m\to\infty} N(a_1,\ldots,a_{n-1},x_k-x_m,t)=1 \tag{3.2}$$

for all $a_1,\ldots,a_{n-1}\in X$ and all $t>0$.

It should be noted that the locally convex topology generated by a fuzzy n-norm is not metrizable, in general. Therefore, in many cases it will be necessary to consider nets $\{x_i\}$ in place of sequences. Of course, Theorems 3.2 and 3.3 generalize in an obvious way to nets.

§4. Fuzzy n-norms on finite dimensional spaces

In this section (X,N) is a fuzzy n-normed space and X has finite dimension at least n. Since the locally convex topology generated by N is Hausdorff by Theorem 3.1. Tihonov's theorem [2, Theorem (23.1)] implies that this locally convex topology is the only one on X. Therefore, all fuzzy n-norms on X are equivalent in the sense that they generate the same locally convex topology.

In the rest of this section we will give a direct proof of this fact (without using Tihonov's theorem). We will set $X=\mathbb{R}^d$ with $d\geqslant n$.

Lemma 4.1 *Every n-seminorm on $X=\mathbb{R}^d$ is continuous as a function on X^n with the euclidian topology.*

Proof For every $j=1,2,\ldots,n$, let $\{x_{j,k}\}_{k=1}^{\infty}$ be a sequence in X converging to $x_j\in X$. Therefore, $\lim\limits_{k\to\infty}\|x_{j,k}-x_j\|=0$, where $\|x\|$ denotes the euclidian norm of x. From property (S4) of an n-seminorm we get

$$|\|x_{1,k},x_{2,k},\ldots,x_{n,k}\|-\|x_1,x_{2,k},\ldots,x_{n,k}\||\le\|x_{1,k}-x_1,x_{2,k},\ldots,x_{n,k}\|.$$

Expressing every vector in the standard basis of $\mathbb{R}^d$ we see that there is a constant M such that

$$\|y_1,y_2,\ldots,\ y_n\|\le M\,\|y_1\|\ldots\|y_n\|\ \text{ for all } y_j\in X.$$

Therefore,

$$\lim_{k\to\infty}\ \|x_{1,k}-x_1,\ x_{2,k},\ldots,\ x_{n,k}\|=0$$

and so

$$\lim_{k\to\infty}\ |\|x_{1,k},\ x_{2,k},\ldots,\ x_{n,k}\|-\|x_1,\ x_{2,k},\ldots,\ x_{n,k}\||=0.$$

We continue this procedure until we reach

$$\lim_{k\to\infty}\|x_{1,k},\ x_{2,k},\ldots,\ x_{n,k}\|=\|x_1,\ x_2,\ldots,\ x_n\|\,.$$

□

Lemma 4.2 *Let $(\mathbb{R}^d, N)$ be a fuzzy n-normed space. Then $\|x_1, x_2, \ldots, x_n\|_\alpha$ is an n-norm if $\alpha \in (0,1)$ is sufficiently close to 1.*

Proof We consider the compact set

$$S = \left\{(x_1,\ x_2, \ldots,\ x_n) \in \mathbb{R}^{dn} : x_1,\ x_2, \ldots,\ x_n \text{ is an orthonormal system in } \mathbb{R}^d\right\}.$$

For each $\alpha \in (0,1)$ consider the set

$$S_\alpha = \{(x_1, x_2, \ldots, x_n) \in S : \|x_1, x_2, \ldots, x_n\|_\alpha > 0\}.$$

By Lemma 4.1, S_α is an open subset of S. We now show that

$$S = \bigcup_{\alpha \in (0,1)} S_\alpha. \tag{4.1}$$

If $(x_1,\ x_2, \ldots,\ x_n) \in S$ then $(x_1,\ x_2, \ldots,\ x_n)$ is linearly independent and therefore there is β such that $N(x_1, x_2, \ldots, x_n, 1) < \beta < 1$. This implies that $\|x_1,\ x_2, \ldots,\ x_n\|_\beta \geq 1$ so (4.1) is proved. By compactness of S, we find $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m$ such that

$$S = \bigcup_{i=1}^{m} S_{\alpha_i}.$$

Let $\alpha = \max\{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m\}$. Then $\|x_1,\ x_2, \ldots,\ x_n\|_\alpha > 0$ for every $(x_1,\ x_2, \ldots,\ x_n) \in S$.

Let $x_1,\ x_2, \ldots,\ x_n \in X$ be linearly independent. Construct an orthonormal system $e_1,\ e_2, \ldots,\ e_n$ from $x_1,\ x_2, \ldots,\ x_n$ by the Gram-Schmidt method. Then there is $c > 0$ such that

$$\|x_1,\ x_2, \ldots,\ x_n\|_\alpha = c\,\|e_1,\ e_2, \ldots,\ e_n\|_\alpha > 0.$$

This proves the lemma. □

Theorem 4.1 *Let N be a fuzzy n-norm on $\mathbb{R}^d$, and let $\{x_k\}$ be a sequence in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$.*

(a) $\{x_k\}$ converges to x with respect to N if and only if $\{x_k\}$ converges to x in the euclidian topology.

(b) $\{x_k\}$ is a Cauchy sequence with respect to N if and only if $\{x_k\}$ is a Cauchy sequence in the euclidian metric.

Proof (a) Suppose $\{x_k\}$ converges to x with respect to euclidian topology. Let $a_1, a_2, \ldots, a_{n-1} \in X$. By Lemma 4.1, for every $\alpha \in (0,1)$,

$$\lim_{k\to\infty} \|a_1,\ a_2, \ldots,\ a_{n-1},\ x_k - x\|_\alpha = 0.$$

By definition of convergence in $(\mathbb{R}^d, N)$, we get that $\{x_k\}$ converges to x in $(\mathbb{R}^d, N)$. Conversely, suppose that $\{x_k\}$ converges to x in $(\mathbb{R}^d, N)$. By Lemma 4.2, there is $\alpha \in (0,1)$ such that $\|y_1, y_2, \ldots, y_n\|_\alpha$ is an n-norm. By definition, $\{x_k\}$ converges to x in the n-normed space $(\mathbb{R}^d, \|\cdot\|_\alpha)$. It is known from [8, Proposition 3.1] that this implies that $\{x_k\}$ converges to x with respect to euclidian topology.

(b) is proved in a similar way. □

Theorem 4.2 *A finite dimensional fuzzy n-normed space (X, N) is complete.*

Proof This follows directly from Theorem 3.4. □

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σ-Coloring of the Monohedral Tiling

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Abstract: In this paper we introduce the definition of σ-coloring and perfect σ-coloring for the plane which is equipped by tiling $\Im$. And we investigate the σ-coloring for the r-monohedral tiling.

Key Words: Smarandache k-tiling, coloring, Monohedral tiling

AMS(2000): 05C15

§1. Introduction

For an integer k, a Smarandache k-tiling of the plane is a family of sets called k-tiles that cover the plane without gaps. Particularly, a Smarandache 1-tiling is usually called tiling of the plane [8]. Tilings are known as tessellations or pavings, they have appeared in human activities since prehistoric times. Their mathematical theory is mostly elementary, but nevertheless it contains a rich supply of interesting problems at various levels. The same is true for the special class of tiling called tiling by regular polygons [2]. The notions of tiling by regular polygons in the plane is introduced by Grunbaum and Shephard in [3]. For more details see [4, 5, 6, 7].

Definition 1.1 *A tiling of the plane is a collection* $\Im = \{T_i : i \in I = \{1, 2, 3, ...\}\}$ *of closed topological discs (tiles) which covers the Euclidean plane* E^2 *and is such that the interiors of the tiles are disjoint.*

More explicitly, the union of the sets $T_1, T_2, ...,$ tiles, is to be the whole plane, and the interiors of the sets T_i topological disc it is meant a set whose boundary is a single simple closed curve. Two tiles are called adjacent if they have an edge in common, and then each is called an adjacent of the other. Two distinct edges are adjacent if they have a common endpoint. The word incident is used to denote the relation of a tile to each of its edges or vertices, and also of an edge to each of its endpoints. Two tilings $\Im_1$ and $\Im_2$ are congruent if $\Im_1$ may be made to coincide with $\Im_2$ by a rigid motion of the plane, possibly including reflection [1].

Definition 1.2 *A tiling is called edge-to-edge if the relation of any two tiles is one of the following three possibilities:*
(a) they are disjoint, or

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(b) they have precisely one common point which is a vertex of each the of polygons, or
(c) they share a segment that is an edge of each of the two polygons.

Hence a point of the plane that is a vertex of one of the polygons in an edge-to-edge tiling is also a vertex of every other polygon to which it belongs and it is called a vertex of the tiling. Similarly, each edge of one of the polygons, regular tiling, is an edge of precisely one other polygon and it is called an edge of the tiling. It should be noted that the only possible edge-to-edge tilings of the plane by mutually congruent regular convex polygons are the three regular tilings by equilateral triangles, by squares, or by regular hexagons. A portion of each of these three tilings is illustrated in Fig.1.

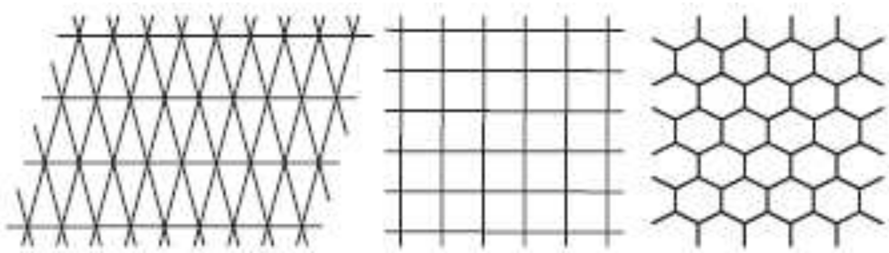

Fig.1

Definition 1.3 *The regular tiling $\Im$ will be called r-monohedral if every tile in $\Im$ is congruent to one fixed set T. The set T is called the prototile of $\Im$, where r is the number of vertices for each tile [2].*

§2. σ-Coloring

Let R^2 be equipped by r-monohedral tiling $\Im$, and let $V(\Im)$ be the set of all vertices of the tiling $\Im$.

Definition 2.1 *σ-coloring of the tiling $\Im$. Is a portion of $V(\Im)$ into k color classes such that:*

(i) different colors appears on adjacent vertices, and for each tile $T_i \in \Im$ there exist permutation σ from some color k.

(ii) The exist at least σ_i such that $O(\sigma_i) = k$. where $O(\sigma_i)$ is the order the permutation σ.

We will denote to the set of all permutation the σ-coloring by $J(\Im)$.

Definition 2.2 *The σ-coloring is called perfect σ-coloring if all tiles have the same permutation.*

Theorem 2.1 *The 3-monohedral tiling admit σ-coloring if and only if $k = 3$.*

Proof Let R^2 be equipped by 3-monohedral tiling $\Im$. If $k < 3$, then there exist adjacent vertices colored by the same color, which it contradicts with condition (i). If $k > 3$, then the

condition (i) satisfied but the condition (ii) not satisfied because as we know that each tile has three vertices, so it cannot be colored by more than three colors, see Fig. 2.

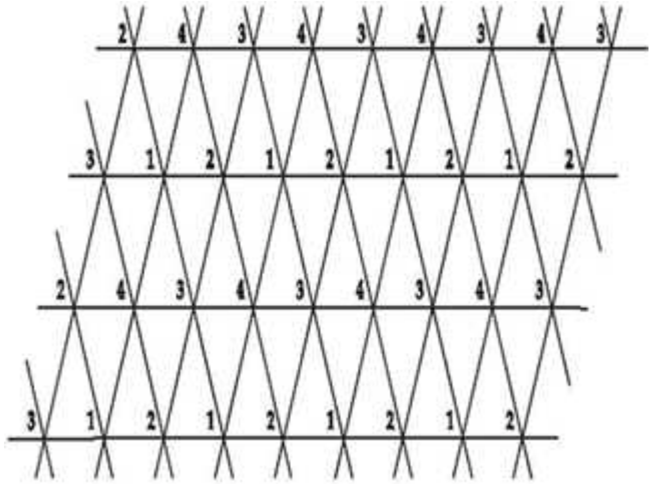

Fig.2

Hence the 3-monohedral tiling admit σ-coloring only if $k = 3$, and σwill be $\sigma = (123)$. see Fig.3.

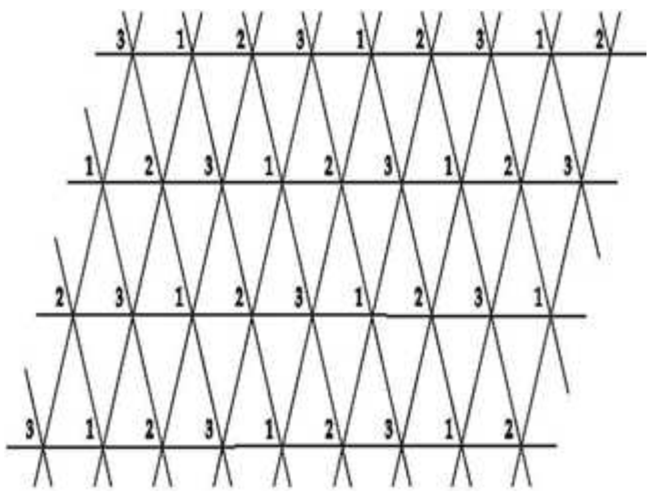

Fig.3

Corollary 2.1 *Every σ-coloring of 3-monohedral tiling is perfect σ-coloring.*

Theorem 2.2 *The 4-monohedral tiling admit σ-coloring if and only if $k = 2$ and $k = 4$.*

Proof Let R^2 be equipped by 4-monohedral tiling $\Im$. If the $k > 4$, then the condition (ii) not satisfied as in the 3-monohedral case, then k have only three cases $k = 1, 2, 3$ and 4. If $k = 1$then there exist adjacent vertices colored by the same color which contradicts this will contradicts with condition (i). If $k = 3$, the first three vertices colored by three colors, so the forth vertex colored by color differ from the color in the adjacent vertices by this way the tiling will be colored but this coloring is not σ-coloring since the permutation from colors not found.

Then the condition (i) not satisfied. If $k = 2$, the two condition of the σ-coloring are satisfied and so the tiling admits σ-coloring by permutation $\sigma = (12)$, for all tiles $T_i \in \Im$, see Fig. 4.

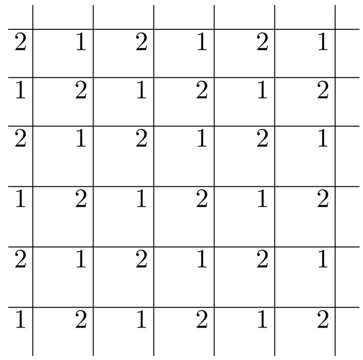

Fig.4

If $k = 4$, then $V(\Im)$ colored by four colors, and in this case the 5-monohedral tiling admit σ-coloring by two methods. The first method that all tiles have the permutation $\sigma = (1234)$, and in this case the σ-coloring will be perfect σ-coloring, see Fig. 5.

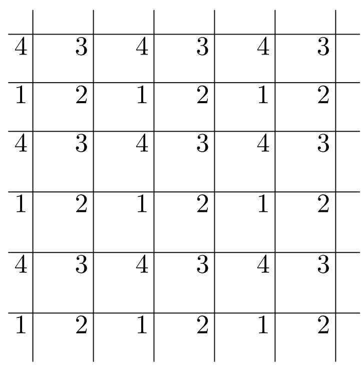

Fig.5

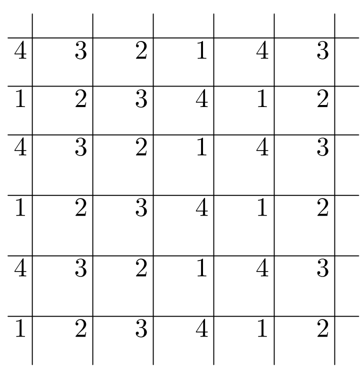

Fig.6

The second method, if the T_i has the permutation $\sigma = (1234)$. As we know, each tile surrounding by eight tiles four tiles adjacent to T_i by edges, and other four tiles adjacent to T_i, by vertices then we can colored these tiles by the four colors such that all tiles have the permutation $\sigma = (1234)$ and the tiles which adjacent by vertices will colored by some colors of k , such that each tile has one of these permutation $\{\alpha, \beta, \gamma, \delta\}$, where $\alpha = (12), \beta = (23), \gamma = (34)$ and $\delta = (41)$, see Figure 6. Then will be $J(\Im) = \{\sigma = (1234), \alpha = (12), \beta = (23), \gamma = (34), \delta = (41)\}$. □

Corollary 2.2 *The 4-monohedral tiling admit prefect σ-coloring if and only if $k = 2$.*

Theorem 2.3 *The 6-monohedral tiling admit σ-coloring if and only if $k = 2, k = 3.$ and $k = 6$.*

Proof If $k > 6$, then the condition (ii) not satisfied as in the 3-monohedral case. Now we will investigate the cases where $k = 1, 2, 3, 4, 5$ and 6. If $k = 1$, then the condition (i) not satisfied as in the 4-monohedral case. If $k = 5$or $k = 4$ we known that each tile has six vertices, then in each case $k = 5$ or $k = 4$ the tiling can be colored by 5 or 4 colors but these colors not satisfied the condition (i). Then at $k = 5$or $k = 4$the coloring not be σ-coloring. If $k = 2$, the tiling coloring by two color then the vertices of each tile colored by two color, and the permutation will be $\sigma = (12)$ for all tiles see Fig. 7.

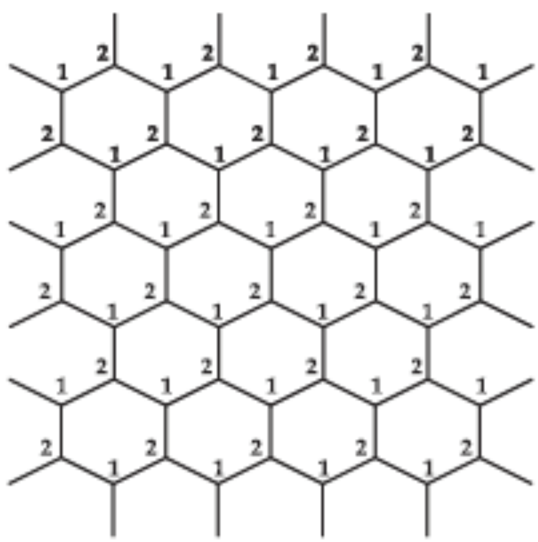

Fig.7

If $k = 3$, the tiling coloring by three colors. So suppose that T_i tile colored by $\sigma = (123)$, we know that each tile surrounding by six tiles, then if the tile which lies on edge $e_1 = (v_1 v_2) \in T_i$ has the permutation $\sigma = (123)$ thus the tiles which lies on $e_4 = (v_4 v_5)$ will colored by $\alpha = (12)$ and the converse is true. Similarly the edges $\{e_2, e_5\}$with the with the permutation $\{\sigma = (123456), \beta = (34)\}$ and $\{e_3, e_6\}$ with $\{\sigma = (123456), \delta = (56)\}$. Then permutation $\{\sigma = (123), \beta = (23)\}$ and $\{e_3, e_6\}$ with $\{\sigma = (123), \delta = (13)\}$, then $J(\Im) = \{\sigma = (123), \alpha = (12), \beta = (23), \delta = (13)\}$. see Fig. 8.

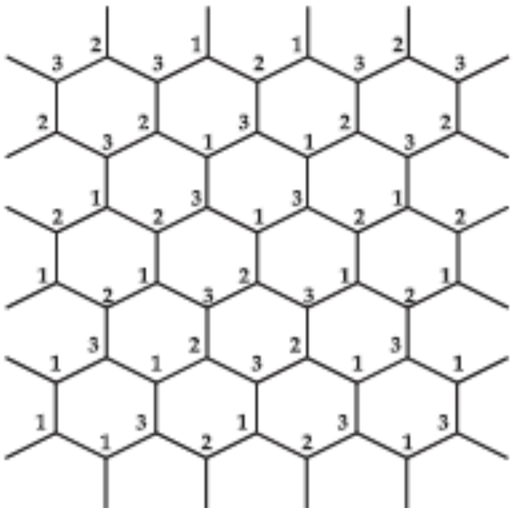

Fig.8

If $k = 6$. the tiling coloring by six colors. Then if the tile which lies on edge $e_1 = (v_1v_2) \in T_i$ has the permutation $\sigma = (123456)$ thus the tiles which lies on $e_4 = (v_4v_5)$ will colored by $\alpha = (12)$ and the converse is true. Similarly the edges $\{e_2, e_5\}$ $J(\Im) = \{\sigma = (123456), \alpha = (12), \beta = (34), \delta = (56)\}$, see Fig. 9. $\square$

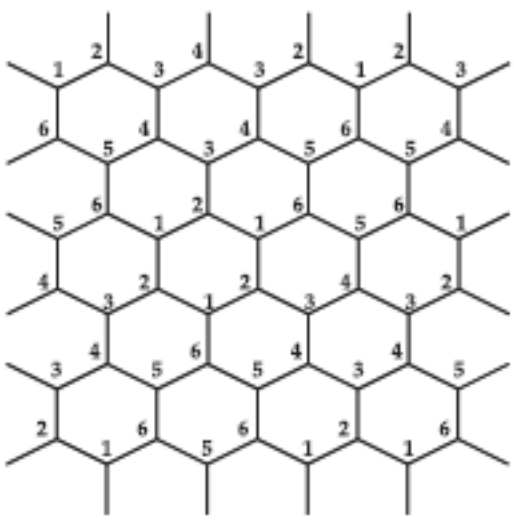

Fig.9

Corollary 2.3 *The 6 -monohedral tiling admit prefect σ-coloring if and only if $k = 2$.*

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The Forcing Domination Number of Hamiltonian Cubic Graphs

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Abstract: A set of vertices S in a graph G is called to be a Smarandachely dominating k-set, if each vertex of G is dominated by at least k vertices of S. Particularly, if $k = 1$, such a set is called a dominating set of G. The Smarandachely domination number $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely dominating set of G. For abbreviation, we denote $\gamma_1(G)$ by $\gamma(G)$. In 1996, Reed proved that the domination number $\gamma(G)$ of every n-vertex graph G with minimum degree at least 3 is at most $3n/8$. Also, he conjectured that $\gamma(H) \geq \lceil n/3 \rceil$ for every connected 3-regular n-vertex graph H. In [**?**], the authors presented a sequence of Hamiltonian cubic graphs whose domination numbers are sharp and in this paper we study forcing domination number for those graphs.

Key Words: Smarandachely dominating k-set, dominating set, forcing domination number, Hamiltonian cubic graph.

AMS(2000): 05C69

§1. Introduction

Throughout this paper, all graphs considered are finite, undirected, loopless and without multiple edges. We refer the reader to [12] for terminology in graph theory.

Let G be a graph, with n vertices and e edges. Let $N(v)$ be the set of neighbors of a vertex v and $N[v] = N(v) \cup \{v\}$. Let $d(v) = |N(v)|$ be the degree of v. A graph G is $r-$regular if $d(v) = r$ for all v. Particularly, if $r = 3$ then G is called a cubic graph. A vertex in a graph G dominates itself and its neighbors. A set of vertices S in a graph G is called to be a *Smarandachely dominating k-set*, if each vertex of G is dominated by at least k vertices of S. Particularly, if $k = 1$, such a set is called a *dominating set* of G. The *Smarandachely domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely dominating set of G. For abbreviation, we denote $\gamma_1(G)$ by $\gamma(G)$. A subset F of a minimum dominating set S is a *forcing subset* for S if S is the unique minimum dominating set containing F. The *forcing domination number* $f(G, \gamma)$ of S is the minimum cardinality among the forcing subsets of S , and the forcing domination number $f(G, \gamma)$ of G is the minimum forcing domination number among

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the minimum dominating sets of G ([1], [2], [5]-[7]). For every graph G, $f(G,\gamma) \le \gamma(G)$. Also The forcing domination number of several classes of graphs are determined, including complete multipartite graphs, paths, cycles, ladders and prisms. The forcing domination number of the cartesian product G of k copies of the cycle C_{2k+1} is studied.

The problem of finding the domination number of a graph is NP-hard, even when restricted to cubic graphs. One simple heuristic is the greedy algorithm, ([11]). Let d_g be the size of the dominating set returned by the greedy algorithm. In 1991 Parekh [9] showed that $d_g \le n+1-\sqrt{2e+1}$. Also, some bounds have been discovered on $\gamma(G)$ for cubic graphs. Reed [10] proved that $\gamma(G) \le \frac{3}{8}n$. He conjectured that $\gamma(H) \ge \lceil \frac{n}{3} \rceil$ for every connected 3-regular (cubic) n-vertex graph H. Reed's conjecture is obviously true for Hamiltonian cubic graphs. Fisher et al. [3]-[4] repeated this result and showed that if G has girth at least 5 then $\gamma(G) \le \frac{5}{14}n$. In the light of these bounds on γ, in 2004 Seager considered bounds on d_g for cubic graphs and showed that ([11]):

For any graph of order n, $\left\lceil \frac{n}{1+\Delta G} \right\rceil \le \gamma(G)$ (see [4]) and for a cubic graph G, $d_g \le \frac{4}{9}n$.

In this paper, we would like to study the forcing domination number for Hamiltonian cubic graphs. In [8], the authors showed that:

Lemma A. If $r \equiv 2$ or $3 \pmod 4$, then $\gamma\left(G^{'}\right) = \gamma(G)$.

Lemma B. If $r \equiv 0$ or $1 \pmod 4$, then $\gamma\left(G^{'}\right) = \gamma(G) - 1$.

Theorem C. If $r \equiv 1 \pmod 4$, then $\gamma(G_0) = m\lceil \frac{n}{4} \rceil - \lceil \frac{m}{3} \rceil$.

§2. Forcing domination number

Remark 2.1 Let $G = (V,E)$ be the graph with $V = \{v_1, v_2, ..., v_n\}$ for $n = 2r$ and $E = \{v_iv_j |\ |i-j| = 1 \text{ } or \text{ } r\}$. So G has two vertices v_1 and v_n of degree two and $n-2$ vertices of degree three. By the graph G is the graph described in Fig.1.

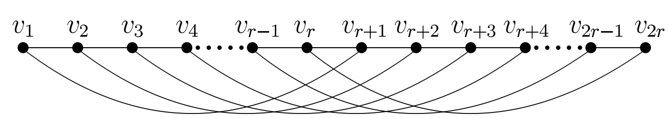

Fig.1. The graph G.

For the following we put $N_p[x] = \{z|\ z \text{ } is \text{ } only \text{ } dominated \text{ } by \text{ } x\} \cup \{x\}$.

Remark 2.2 Suppose that the graphs $G^{'}$ and $G^{''}$ are two induced subgraphs of G such that $V(G^{'}) = V(G) - \{v_1, v_n\}$ and $V(G^{''}) = V(G) - \{v_1\}$ (or $V(G^{''}) = V(G) - \{v_{2r}\}$).

Remark 2.3 Let G_0 be a graph of order mn that $n = 2r$, $V(G_0) = \{v_{11}, v_{12}, ...,$ $v_{1n}, v_{21}, v_{22}, ..., v_{2n}, ..., v_{m1}, v_{m2} ..., v_{mn}\}$ and $E = \cup_{i=1}^{m}\{v_{ij}v_{il}|\ |j-l| = 1 \text{ } or \text{ } r\} \cup \{v_{in}v_{(i+1)1}|\ i = 1, 2, ..., m-1\} \cup \{v_{11}v_{mn}\}$. By the graph G_0 is 3-regular graph. Suppose that the graph G_i

is an induced subgraph of G_0 with the vertices $v_{i1}, v_{i1}, ..., v_{in}$. By the graph G_0 is the graph described in Fig. 2.

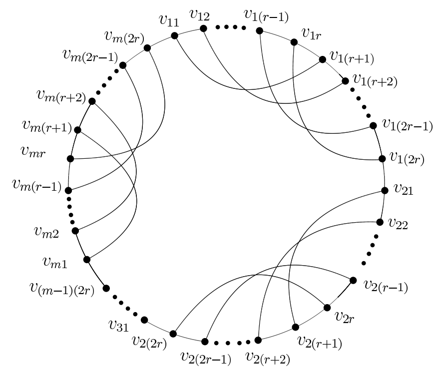

Fig. 2. The graph G_0.

Proposition 2.4 *If* $r \equiv 0$ *(mod 4), then* $f(G, \gamma) \leq 2$, *otherwise* $f(G, \gamma) = 1$.

proof First we suppose that $r \equiv 1$ (mod 4). It is easy to see that $f(G, \gamma) > 0$, because G has at least two minimum dominating set. Suppose $F = \{v_1\} \subset S$ where S is a minimum dominating set. Since $\gamma(G) = 2\lfloor r/4 \rfloor + 1$, for two vertices v_x and v_y in S, $|N[v_x] \cup N[v_y]| \geq 6$. This implies that $\{v_2, v_{r+1}\} \cap S = \emptyset$, then $v_{r+3} \in S$. A same argument shows that $v_5 \in S$. Thus S must be contains $\{v_{r+7}, v_9, ..., v_{2r-2}, v_r\}$, therefore $f(G, \gamma) = 1$.

If $r \equiv 2$ (mod 4), we consider $S = \{v_2, v_6, v_{10}, ..., v_r, v_{r+4}, v_{r+8}, ..., v_{2r-6}, v_{2r-2}\}$. Assign the set $F = \{v_2\}$ then it follows $f(G, \gamma) \leq 1$, because $|N_p[x]| = 4$ to each vertex $x \in S$. On the other hand since G has at least two minimum dominating set. Hence $f(G, \gamma) = 1$.

If $r \equiv 3$ (mod 4), for $S = \{v_1, v_5, v_9, ..., v_{r-2}, v_{r+3}, v_{r+7}, ..., v_{2r-4}, v_{2r}\}$, the set $F = \{v_1\}$ shows that $f(G, \gamma) \leq 1$. Further, since G has at least two minimum dominating set, then it follows $f(G, \gamma) = 1$.

Finally let $r \equiv 0$ (mod 4), we consider $S = \{v_1, v_5, v_9, ..., v_{r-3}, v_{r+1}, v_{r+3}, v_{r+7}, ..., v_{2r-5}, v_{2r-1}\}$. If $F = \{v_1, v_{r+1}\}$, a simple verification shows that $f(G, \gamma) \leq 2$. □

Proposition 2.5 *If* $r \equiv 1$ *(mod 4) then* $f\left(G', \gamma\right) = 0$.

Proof By Lemma B, we have $\gamma\left(G'\right) = 2\lfloor r/4 \rfloor$. Now, we suppose that S is an arbitrary minimum dominating set for G'. Obviously for each vertex $v_x \in S$, $|N_p[v_x]| = 4$, so $\{v_{r-1}, v_{r+2}\} \subset S$. But $\{v_{2r-2}, v_{r-2}\} \cap S = \emptyset$ therefore $v_{2r-3} \in S$. Thus S must be contains $\{v_{r-5}, v_{r-9}, ..., v_{r+10}, v_{r+6}\}$, then S is uniquely determined and it follows that $f\left(G', \gamma\right) = 0$.□

Proposition 2.6 *If* $r \equiv 0$ *(mod 4) then* $f\left(G'', \gamma\right) = 0$.

Proof Let $r \equiv 0$ (mod 4) and S be an arbitrary minimum dominating set for G'' with $V(G'') = V(G) - \{v_1\}$. If $\{v_{2r}, v_{2r-1}\} \cap S \neq \emptyset$. Without loss of generality, we assume that $v_{2r} \in S$ then S must be contains $\{v_{r+2}, v_{r-2}, v_{r-6}, ..., v_{10}, v_6, v_{2r-4}, v_{2r-8}, ..., v_{r+8}\}$. On the other hand by Lemma B, $\gamma\left(G''\right) = 2\lfloor r/4 \rfloor$ (Note that by Proof of Lemma B one can see

$\gamma(G^{'})=\gamma(G^{''})$ where $r\equiv 0 \pmod 4$). So the vertices v_3, v_4, v_{r+4} and v_{r+5} must be dominated by one vertex and this is impossible. Thus necessarily $v_r \in S$, but $\{v_{r-1}, v_{2r-1}\}\cap S=\emptyset$ which implies $v_{2r-2}\in S$. Finally the remaining non-dominated vertices $\{v_{r+1}, v_{r+2}, v_2\}$ is just dominated by v_{r+2}. Therefore the set $S=\{v_4, v_8, ..., v_{r-4}, v_r, v_{r+2}, v_{r+6}, ..., v_{2r-2}\}$ is uniquely determined which implies $f\left(G^{''},\gamma\right)=0$. □

§3. Main Results

Theorem 3.1 *If $r\equiv 2$ or 3 (mod 4), then $f(G_0,\gamma)=m$.*

Proof Let $r\equiv 2 \pmod 4$ and S be a minimum dominating set for G_0. If there exists $i\in\{1,2,...,m\}$ such that $S\cap\{v_{i1}, v_{in}\}\neq\emptyset$ then it implies $|S\cap G_i| > 2\lfloor r/4\rfloor+1$. Moreover $\gamma(G_0)=m(2\lfloor r/4\rfloor+1)$. From this it immediately follows that there exists $j\in\{1,2,...,m\}-\{i\}$ such that $|S\cap G_j|<2\lfloor r/4\rfloor+1$ and this is contrary to Lemma A. Hence $S\cap\{v_{i1}, v_{in}\}=\emptyset$ for $1\le i\le m$. On the other hand $f(G_i,\gamma)=1$ for $1\le i\le m$ which implies $f(G_0,\gamma)=m$.

Now we suppose that $r\equiv 3 \pmod 4$ and S is minimum dominating set for G_0, such that $F=\{v_{i1}|\,1\le i\le m\}\subset S$. Since $v_{i1}\in S$ and $\gamma(G_0)=2\lfloor r/4\rfloor+2$ then $\{v_{i2}, v_{i3}\}\cap S=\emptyset$ and this implies $v_{i(r+3)}\in S$. With similar description, we have $\{v_{i5}, v_{i9}, ..., v_{i(r-2)}, v_{i(r+6)}, v_{i(r+11)}, ..., v_{i(2r-4)}\}\subset S$. But for the remaining non-dominated vertices $v_{ir}, v_{i(2r)}$ and $v_{i(2r-1)}$ necessarily implies that $v_{i(2r)}\in S$. Hence S is the unique minimum dominating set containing F. Thus $f(G_0,\gamma)\le m$. A trivial verification shows that $f\left(G^{'},\gamma\right), f\left(G^{''},\gamma\right)\ge 1$ for $i\in\{1,2,...,m\}$, therefore $f(G_0,\gamma)=m$. □

Theorem 3.2 $f(G_0,\gamma)=\begin{cases} 1 & \textit{if } m\equiv 0 \ (mod\ 3) \\ 2 & \textit{otherwise} \end{cases}$ *for $r\equiv 1$ (mod 4).*

Proof If $m\equiv 0 \pmod 3$, we suppose that $F=\{v_{1n}\}\subset S$ and S is a minimum dominating set for G_0. By Theorem C, we have $\gamma(G_0)=m\lceil n/4\rceil-\lfloor m/3\rfloor$, then $v_{3,1}\in S$. Here, we use the proof of Propositions 4 and 5. From this the sets $S\cap V(G_1)$, $S\cap V(G_2)$, $S\cap V(G_3)$ uniquely characterize. By continuing this process the set S uniquely obtain, then $f(G_0,\gamma)=1$.

If $m\equiv 1$ or $2 \pmod 3$, then the set $F=\{v_{1n}, v_{mn}\}$ uniquely characterize the minimum dominating set for G_0, therefore $f(G_0,\gamma)=2$. □

Theorem 3.3 $f(G_0,\gamma)=\begin{cases} \left\lfloor \frac{m}{3}\right\rfloor+1 & \textit{if } m\equiv 0 \ (mod\ 3) \\ \left\lfloor \frac{m}{3}\right\rfloor+3 & \textit{otherwise} \end{cases}$ *for $r\equiv 0$ (mod 4).*

Proof If $m\equiv 0 \pmod 3$ the set $F=\{v_{21}, v_{2(r+4)}, v_{5(r+4)}, v_{8(r+4)}, ..., v_{m-1(r+4)}\}$ determine the unique minimum dominating set for G_0 then $f(G_0,\gamma)\le\lfloor m/3\rfloor+1$. But $\gamma(G_i)=2\lfloor r/4\rfloor$ for $\lfloor m/3\rfloor$ of G_is. Hence $f(G_0,\gamma)=\lfloor m/3\rfloor+1$. The proof of the case $m\equiv 1$ or $2 \pmod 3$ is similar to the previous case. □

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Permutation Polynomials modulo n, $n \neq 2^w$ and Latin Squares

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Abstract: Our work is motivated by a recent paper of Rivest [6], concerning permutation polynomials over the rings Z_n with $n = 2^w$. Permutation polynomials over finite fields and the rings Z_n have lots of applications, including cryptography. For the special case $n = 2^w$, a characterization has been obtained in [6] where it is shown that such polynomials can form a Latin square $(0 \leq x, y \leq n-1)$ if and only if the four univariate polynomials $P(x,0), P(x,1), P(0,y)$ and $P(1,y)$ are permutation polynomials. Further, it is shown that pairs of such polynomials will never form Latin squares. In this paper, we consider bivariate polynomials $P(x,y)$ over the rings Z_n when $n \neq 2^w$. Based on preliminary numerical computations, we give complete results for linear and quadratic polynomials. Rivest's result holds in the linear case while there are plenty of counterexamples in the quadratic case.

Key Words: Permutation polynomials, Latin squares, Orthogonal Latin squares, Orthomorphisms.

AMS(2000): 05B15

§1. Permutation Polynomials

A polynomial $P(x) = a_0 + a_1x + ... + a_dx^d$ is said to be a permutation polynomial over a finite ring R if P permutes the elements of R. R. Lidl and H. Niderreiter [2] have described various types of permutation polynomials over finite fields F_q. Lidl and Mullen [3], [4] gave a survey of various possibilities of polynomials over finite fields as permutation polynomials and also gave the applications of these permutation polynomials. Rivest [6] has considered the class of rings Z_n, where $n = 2^w$ to study the permutaion polynomials. He derived necessary and sufficient conditions for a polynomial to be a permutation polynomial over Z_n, where $n = 2^w$, in terms of the coefficients of the polynomials. The following is from [6]:

Theorem 1(Rivest) *Let $P(x) = a_0 + a_1x + ... + a_dx^d$ be a polynomial with integral coefficients. Then $P(x)$is a permutation polynomial modulo $n = 2^w$, $w \geq 2$, if and only if a_1 is odd and both $(a_2 + a_4 + ...)$ and $(a_3 + a_5 + ...)$ are even.*

Also, Rivest gave a result about bivariate polynomials $P(x, y)$ giving latin squares modulo

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$n = 2^w$,$w \geq 2$. The following result is also from [6]:

Theorem 2(Rivest) *A bivariate polynomial $P(x, y) = \sum_{i,j} a_{ij} x^i y^j$ represents a Latin square modulo $n = 2^w$, where $w \geq 2$, if and only if the four univariate polynomials $P(x, 0), P(x, 1), P(0, y)$ and $P(1, y)$ are all permutation polynomials modulo n.*

§2. **Latin squares**

A Latin square of order n is an $n \times n$ array based on some set S of n symbols, with the property that every row and every column contains every symbol exactly once. In other words, every row and every column is a permutation of S. Since the arithmetical properties of symbols are not used, the nature of the elements of S is immaterial. An example of a Latin square of order 4 is shown below.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

Two Latin squares A and B of the same order are said to be equivalent if it is possible to reorder the rows of A, reorder the columns of A, and/or relabel the symbols of A in such a way as to produce the square B. A partial Latin square of order n is an $n \times n$ array in which some cells are filled with the elements of some n-set while others are empty, such that no row or column contains a repeated element. A Latin rectangle of size $k \times n$ is a $k \times n$ array with entries from $S = \{0, 1, 2...., n-1\}$ such that every row is a permutation of S and the columns contain no repetitions.
The following theorem is proved in [7]:

Theorem 3 *If A is a $k \times n$ Latin rectangle, then one can append $(n-k)$ further rows to A so that the resulting array is a Latin square.*

If L is Latin square of order s and $n \geq 2s$, then there is a Latin square of order n with L as a subsquare [7]. Starting from a partial Latin square of order n, it is possible to complete it to a Latin square of order n, see [5].

Theorem 4 *A partial Latin square of order n with at most $n-1$ filled cells can be completed to a Latin square of order n.*

Two Latin squares of order n are called orthogonal if each of the n^2 ordered pairs $(0, 0), ., (n-1, n-1)$ appears exactly once in the two squares. A pair of orthogonal Latin squares of order 4 is shown below.

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

A Latin square is called self-orthogonal if it is orthogonal to its own transpose.Latin squares and orthogonal Latin squares have been extensively studied since Euler considered it first in 1779. Euler knew that a pair of orthogonal Latin squares of order n existed for all odd values of n and all $n \equiv 0(mod4)$. Euler went on to assert that no such pairs exist for $n \equiv 2(mod4)$, this was known as *Euler's conjecture* for 177 years until it was suddenly and completely disproved by Bose, Parker and Shrikhande. Indeed, the only exceptions are $n = 2, 6$ and for all other values, pairs of orthogonal Latin squares exist [5]. Recently, G. Appa, D. Magos, I. Mourtos gave an LP-based proof that there is no pair of orthogonal Latin squares of order 6 (see [1]).

Rivest [6] considered such polynomials *modulo* $n = 2^w$, where $w \geq 2$ and showed that orthogonal pairs of Latin squares do not exist [6]. Here we have considered them modulo n, $n \neq 2^w$ and to our surprise, found that there are many examples of orthogonal pairs of Latin squares. Based on preliminary computations, if $n \neq 2^w$, we have found that a bivariate polynomial can fail to form a Latin square even when these 4 univariate polynomials are permutation polynomials. In a Latin square determined by $P(x, y)$, values of $P(x, 0)$, $P(x, 1)$, $P(0, y)$ and $P(1, y)$ are given by the entries of first two columns and first two rows.

Theorem 5 *A bivariate linear polynomial $a + bx + cy$ represents a latin square over Z_n if and only if one of the following equivalent conditions is satisfied:*

(i) both b and c are coprime with n;

(ii) $a + bx$, $a + cy$, $(a + c) + bx$ and $(a + b) + cy$ are all permutation polynomials modulo n.

Proof For linear polynomials over any Z_n,we can observe that $a + bx + cy$ forms a Latin square if and only if $a + bx$, $a + cy$, $(a + c) + bx$, $(a + b) + cy$ are permutation polynomials. This is because, whenever b and c are both co-prime with n, all those 4 polynomials will be permutation polynomials and in those cases we can fill all the entries of the Latin squares by just looking at first row and first column. As these are all distinct elements in the first row and column, and polynomial $bx + cy$ having only two terms, the entries are got by just adding $a(mod\ n)$ to all entries of $bx + cy$. So Rivest's result holds in the linear case. □

Quadratic case: We also tried to characterize quadratic bivariate polynomials in this way. If a polynomial $P(x, y)$ represents a Latin square, then our 4 polynomials $P(x, 0)$, $P(x, 1)$, $P(0, y)$ and $P(1, y)$ will be obviously permutation polynomials, as they form the first two rows and first two columns of the Latin squares. However, to our surprise, many quadratic polynomials failed to form Latin squares, even though the 4 polynomials $P(x, 0)$, $P(x, 1)$, $P(0, y)$ and $P(1, y)$ are permutation polynomials. The number of such polynomials over different rings Z_n are shown below.

Ring	**No. of polynomials**	**Examples**
Z_6	48	$1+5x+2y+2xy+3y^2$
Z_7	1,050	$x+y+xy$
Z_9	4,374	$x+y+xy+3y^2$
Z_{10}	1,440	$9x+9y+8xy$
Z_{11}	8,910	$10x+10y+10xy$
Z_{12}	768	$7x+7y+10xy+6x^2+6y^2$
Z_{13}	1,8876	$12x+12y+12xy$
Z_{14}	8,400	$13x+11y+6xy$
Z_{15}	3,720	$8x+14y+14xy$

However, there are plenty of quadratic bivariate polynomials which do form Latin squares. But we are not able to characterize them using the permutation behavior of the corresponding univariate polynomials. From the data collected, we observed that in all cases where $P(x,y)$ formed a Latin square, the cross term xy was always absent. Hence we could formulate and prove two interesting results.

However, we need an interesting fact regarding orthomorphisms in proving the theorem. The definition as well as proof of the theorem quoted are given in the well-known text of J.H.Van Lint and R.M.Wilson, *A Course in Combinatorics*, chapter 22, page 297.

Definition 2.1 *An* orthomorphism *of an abelian group G is a permutation σ of the elements of G such that $x \mapsto \sigma(x) - x$ is also a permutation of G.*

Theorem 6 *If an abelian group G admits an orthomorphism, then its order is odd or its Sylow 2-subgroup is not cyclic.*

We are now ready to state and prove the main results of this paper:

Theorem 7 *If $P(x,y)$ is a bivariate polynomial having* no cross term*, then $P(x,y)$ gives a Latin square if and only if $P(x,0)$ and $P(0,y)$ are permutation polynomials.*

Proof $P(x,0)$ is the first column of the square and $P(0,y)$ is the first row. If $P(x,y) = f(x)+g(y)$, looking at first row and column, we can complete the square just as addition modulo n (which is a group). So, $P(x,y)$ will be a Latin square. □

Theorem 8 *Let n be even and $P(x,y) = f(x)+g(y)+xy$ be a bivariate quadratic polynomial, where $f(x)$ and $g(x)$ are permutation polynomials modulo n. Then $P(x,y)$ does not give a Latin square modulo n.*

Proof We assume that n is even and greater than 2. If $f(x)$ is a permutation polynomial then $f(x)+k$ is also a permutation polynomial. So, we can assume that $k = 0$.Now $f(x)+g(y)$ always represents a Latin square whenever $f(x)$ and $g(y)$ are permutation polynomials, by the last theorem. When $x = c$, the cth row entries will be $P(c,0), P(c,1), ...P(c,n-1)$. i.e., $f(c)+$

$g(0)+0, f(c)+g(1)+c, f(c)+g(2)+2c, ..., f(c)+g(n-1)+(n-1)c$ Let $f(c)=\theta$, a constant. Then, $\theta+0, \theta+c, ..., \theta+(n-1)c$ will be a permutation of $\{0,1, ..., n-1\}$ if $g.c.d.(n,c)=1$. So, let c be such that $g.c.d.(n,c)=1$ Without loss of generality, we may ignore the constant θ in the sequence.Also $g(0), g(1), ..., g(n-1)$ is some permutation of $\{0,1,\ldots,n-1\}$. The sum of these two permutations fails to be a permutation of Z_n, since there are no orthomorphisms of Z_n as n is even. Hence the cth row contains repetitions and $P(x,y)$ does not represent a Latin square. □

In case of some bivariate polynomials, the resulting squares will not be Latin squares. But we can get a Latin square of lower order by deleting some rows and columns in which entries have repetitions.Obviously, number of rows and columns deleted must be equal. For example, the polynomial $5x+2y+2xy+3y^2$ over Z_6 will not form a latin square as shown below.

0	5	4	3	2	1
5	0	1	2	3	4
4	1	4	1	4	1
3	2	1	0	5	4
2	3	4	5	0	1
1	4	1	4	1	4

The third and sixth rows as well as columns contain repetitions . In these rows and columns we see only the entries 1 and 4. Deleting these two rows and columns, we get a square of order 4×4, which is a Latin square over the set $\{0,2,3,5\}$.

0	5	3	2
5	0	2	3
3	2	0	5
2	1	5	0

Similarly, the bivariate $P(x,y)=9x+9y+8xy$ over Z_{10} will give a 10×10 square which can be reduced to a Latin square of order 8×8 after deleting 2 rows and 2 columns, having only the entries 3 and 8.

$$P(2,y)=\begin{cases} 3 & \text{for all odd y} \\ 8 & \text{for all even y} \end{cases}$$

$$P(7,y)=\begin{cases} 8 & \text{for all odd y} \\ 3 & \text{for all even y} \end{cases}$$

Similar expressions hold for $P(x,2)$ and $P(x,7)$, because $P(x,y)$ is a symmetric polynomial. So we delete the rows and columns corresponding to both x and y equal to 2 and 7.

Rivest [6] proved that no two bivariate polynomials modulo 2^w, for $w \geq 1$ can form a pair of orthogonal Latin squares. This is because all the bivariate polynomials over Z_n, where $n = 2^w$, will form Latin squares which can be equally divided into 4 parts as shown below, where the $n/2 \times n/2$ squares A and D are identical and $n/2 \times n/2$ squares B and C are identical.

A	B
C	D

So, no two such Latin squares can be orthogonal.

But we do have examples of bivariate polynomials modulo $n \neq 2^w$, such that resulting Latin squares are orthogonal. The two bivariate quadratic polynomials $6x^2+3y^2+3xy+x+5y$ and $3x^2+6y^2+6xy+4x+7y$ give two orthogonal Latin squares over Z_9. Also, $x+4y+3xy$ is a quadratic bivariate which gives a Latin square orthogonal to Latin square formed by $6x^2+3y^2+3xy+x+5y$ over Z_9.

0	8	4	6	5	1	3	2	7
7	0	8	4	6	5	1	3	2
8	4	6	5	1	3	2	7	0
3	2	7	0	8	4	6	5	1
1	3	2	7	0	8	4	6	5
2	7	0	8	4	6	5	1	3
6	5	1	3	2	7	0	8	4
4	6	5	1	3	2	7	0	8
5	1	3	2	7	0	8	4	6

Latin square formed by

$6x^2+3y^2+3xy+x+5y$

0	4	2	3	7	5	6	1	8
7	8	3	1	2	6	4	5	0
2	0	1	5	3	4	8	6	7
3	7	5	6	1	8	0	4	2
1	2	6	4	5	0	7	8	3
5	3	4	8	6	7	2	0	1
6	1	8	0	4	2	3	7	5
4	5	0	7	8	3	1	2	6
8	6	7	2	0	1	5	3	4

Latin square formed by

$3x^2+6y^2+6xy+4x+7y$

We have found many examples in which the rows or columns of the Latin square formed by quadratic bivariates over Z_n are cyclic shifts of a single permutation of $\{0, 1, 2, ..., n-1\}$. If two bivariates give such Latin squares, then corresponding to any one entry in one Latin square, if there are n different entries in n rows of the other Latin square, then those two Latin squares will be orthogonal. For instance, in the above example, the entries in the second square corresponding to the entry 0 in the first square are 0,8,7,6,5,4,3,2,1. The rows of the first square are all cyclic shifts of the permutation (0,8,4,6,5,1,3,2,7), not in order. Also the columns of the second square are the cyclic shifts of the permutation (0,7,2,3,1,5,6,4,8), not in order. We have listed below the number of quadratic bivariates that form Latin squares over Z_n, for $5 \leq n \leq 24$.

n	number of quadratic bivariates (with constant term = 0) forming Latin squares	n	number of quadratic bivariates (with constant term = 0) forming Latin squares
5	16	15	64
6	16	16	32,768
7	36	17	256
8	1,024	18	32,888
9	972	19	324
10	64	20	512
11	100	21	144
12	128	22	400
13	144	23	484
14	144	24	4,096

The following have been noted from the extensive computations carried out on a Personal Computer:

If we write a quadratic bivariate $P(x, y) = a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{02}y^2$, then the numbers in the above table can be explicitly given as the possible choices for the coefficients in $P(x, y)$. We can clearly observe that if $P(x, y)$ forms a Latin square then $P(y, x)$ will form the Latin square which is just a transpose of the former.

In Z_9, there are 972 quadratics with constant term zero, forming Latin squares. These polynomials have the coefficients a_{10} and a_{01} from the set $\{1, 2, 4, 5, 7, 8\}$, coefficients a_{20} and a_{02} from the set $\{0, 3, 6\}$ and the coefficient a_{11} from the set $\{0, 3, 6\}$. So,there are 6 choices for both a_{10} and a_{01}, and 3 choices for each of the coefficients a_{20}, a_{02} and a_{11}. So, the number of such polynomials is equal to $6 \times 6 \times 3 \times 3 \times 3 = 972$. Also we observe that in the case of Z_n where n is a prime or a product of distinct odd primes, the coefficients of x^2, y^2 and xy are all zero. So, in these type of rings we find the number of polynomials that yield Latin squares is k^2, where k is the number of possible coefficients of x and y. When n is a prime number, all $n - 1$ nonzero elements of Z_n occur as coefficients of both x and y. When n is a product of distinct odd primes, then all the $\varphi(n)$ nonzero elements of Z_n which are coprime with n occur as as coefficients of both x and y.

We tabulate a few cases below:

n	**number of P(x,y)**	**set of possible values of a_{10} and a_{01}**
3	$4 = 2^2$	$\{1, 2\}$
5	$16 = 4^2$	$\{1, 2, 3, 4\}$
7	$36 = 6^6$	$\{1, 2, 3, 4, 5, 6\}$
11	$100 = 10^2$	$\{1, 2, ...10\}$

n	**number of P(x,y)**	**set of possible values of a_{10} and a_{01}**
13	$144 = 12^2$	$\{1, 2, ...12\}$
15	$64 = 8^2$	$\{1, 2, 4, 7, 8, 11, 13, 14\}$
17	$256 = 16^2$	$\{1, 2, ...16\}$
19	$324 = 18^2$	$\{1, 2, ...18\}$
21	$144 = 12^2$	$\{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$
23	$484 = 22^2$	$\{1, 2, ...22\}$

From the above table we can see that the number N of bivariate quadratic polynomials $P(x, y)$ with constant term zero which yield Latin squares is given by $N = (\varphi(n))^2$, if n is a prime or product of distinct odd primes.

§3. Conclusion

We have examined Rivest's results when $n \neq 2^w$. A computational study, though on a small scale, has revealed lot of surprises. The bivariate permutation polynomials producing Latin squares do not seem to depend on the behavior of the corresponding univariate polynomials. Several pairs of orthogonal Latin squares are obtained through Latin squares got via permutation polynomials. It would be interesting to know the relation between the coefficients of the polynomials and the relation to the Latin squares and if possible get an expression for their number in terms of the prime decomposition of n. Also, the cubic and higher degrees seem to be much more challenging and will be taken up for later study.

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Graphoidal Tree d - Cover

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Abstract: In [1] Acharya and Sampathkumar defined a graphoidal cover as a partition of edges into internally disjoint (not necessarily open) paths. If we consider only open paths in the above definition then we call it as a graphoidal path cover [3]. Generally, a Smarandache graphoidal tree (k, d)-cover of a graph G is a partition of edges of G into trees $T_1, T_2, \cdots, T_l$ such that $|E(T_i) \cap E(T_j)| \le k$ and $|T_i| \le d$ for integers $1 \le i, j \le l$. Particularly, if $k = 0$, then such a tree is called a graphoidal tree d-cover of G. In [3] a graphoidal tree cover has been defined as a partition of edges into internally disjoint trees. Here we define a graphoidal tree d-cover as a partition of edges into internally disjoint trees in which each tree has a maximum degree bounded by d. The minimum cardinality of such d-covers is denoted by $\gamma_T^{(d)}(G)$. Clearly a graphoidal tree 2-cover is a graphoidal cover. We find $\gamma_T^{(d)}(G)$ for some standard graphs.

Key Words: Smarandache graphoidal tree (k, d)-cover, graphoidal tree d-cover, graphoidal cover.

AMS(2000): 05C70

§1. Introduction

Throughout this paper G stands for simple undirected graph with p vertices and q edges. For other notations and terminology we follow [2]. A Smarandache graphoidal tree (k, d)-cover of G is a partition of edges of G into trees $T_1, T_2, \cdots, T_l$ such that $|E(T_i) \cap E(T_j)| \le k$ and $|T_i| \le d$ for integers $1 \le i, j \le l$. Particularly, if $k = 0$, then such a cover is called a graphoidal tree d-cover of G. A graphoidal tree d-cover $(d \ge 2)$ $\mathscr{F}$ of G is a collection of non-trivial trees in G such that

(i) *Every vertex is an internal vertex of at most one tree;*

(ii) *Every edge is in exactly one tree;*

(iii) *For every tree* $T \in \mathscr{F}, \Delta(T) \le d$.

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Let $\mathscr{G}$ denote the set of all graphoidal tree d-covers of G. Since $E(G)$ is a graphoidal tree d-cover, we have $\mathscr{G} \neq \emptyset$. Let $\gamma_T^{(d)}(G) = \min_{\mathscr{J} \in \mathscr{G}} |\mathscr{J}|$. Then $\gamma_T^{(d)}(G)$ is called the graphoidal tree d-covering number of G. Any graphoidal tree d-cover of G for which $|\mathscr{J}| = \gamma_T^{(d)}(G)$ is called a minimum graphoidal tree d-cover.

A graphoidal tree cover of G is a collection of non-trivial trees in G satisfying (i) and (ii). The minimum cardinality of graphoidal tree covers is denoted by $\gamma_T(G)$. A graphoidal path cover (or acyclic graphoidal cover in [5]) is a collection of non-trivial path in G such that every vertex is an internal vertex of at most one path and every edge is in exactly one path. Clearly a graphoidal tree 2-cover is a graphoidal path cover and a graphoidal tree d-cover ($d \geq \Delta$) is a graphoidal tree cover. Note that $\gamma_T(G) \leq \gamma_T^{(d)}(G)$ for all $d \geq 2$. It is observe that $\gamma_T^{(d)}(G) \geq \Delta - d + 1$.

§2. Preliminaries

Theorem 2.1([4]) $\gamma_T(K_p) = \lceil \frac{p}{2} \rceil$.

Theorem 2.2([4]) $\gamma_T(K_{n,n}) = \lceil \frac{2n}{3} \rceil$.

Theorem 2.3([4]) *If $m \leq n < 2m-3$, then $\gamma_T(K_{m,n}) = \lceil \frac{m+n}{3} \rceil$. Further more, if $n > 2m-3$, then $\gamma_T(K_{m,n}) = m$.*

Theorem 2.4([4]) $\gamma_T(C_m \times C_n) = 3$ *if* $m, n \geq 3$.

Theorem 2.5([4]) $\gamma_T(G) \leq \lceil \frac{p}{2} \rceil$ *if* $\delta(G) \geq \frac{p}{2}$.

§3. Main results

We first determine a lower bound for $\gamma_T(d)(G)$. Define $n_d = \min_{\mathscr{J} \in \mathscr{G}_d} n_{\mathscr{J}}$, where $\mathscr{G}_d$ is a collection of all graphoidal tree d-covers and $n_{\mathscr{J}}$ is the number of vertices which are not internal vertices of any tree in $\mathscr{J}$.

Theorem 3.1 *For $d \geq 2$, $\gamma_T(d)(G) \geq q - (p - n_d)(d-1)$.*

Proof Let Ψ be a minimum graphoidal tree d-cover of G such that n vertices of G are not internal in any tree of Ψ.

Let k be the number of trees in Ψ having more than one edge. For a tree in Ψ having more than one edge, fix a root vertex which is not a pendant vertex. Assign direction to the edges of the k trees in such a way that the root vertex has in degree zero and every other vertex has in degree 1. In Ψ, let l_1 be the number of vertices of out degree d and l_2 the number of vertices of out degree less than or equal to $d-1$ (and > 0) in these k trees. Clearly $l_1 + l_2$ is the number of internal vertices of trees in Ψ and so $l_1 + l_2 = p - n$. In each tree of Ψ there is at most one vertex of out degree d and so $l_1 \leq k$. Hence we have

$$
\begin{aligned}
\gamma_T^{(d)} &\geq k+q-(l_1 d + l_2(d-1)) = k+q-(l_1+l_2)(d-1)l_1 \\
&= k+q-(p-n_\Psi)(d-1)-l_1 \geq q-(p-n_d)(d-1).
\end{aligned}
$$

□

Corollary 3.2 $\gamma_T^{(d)}(G) \geq q-p(d-1)$.

Now we determine graphoidal tree d-covering number of a complete graph.

Theorem 3.3 *For any integer $p \geq 4$,*

$$
\gamma_T^{(d)}(K_p) = \begin{cases} \frac{p(p-2d+1)}{2} & if\ d < \frac{p}{2}; \\ \lceil \frac{p}{2} \rceil & if\ d \geq \frac{p}{2}. \end{cases}
$$

Proof Let $d \geq \frac{p}{2}$. We know that $\gamma_T^{(d)}(K_p) \geq \gamma_T(K_p) = \lceil \frac{p}{2} \rceil$ by Theorem 2.1.

Case (i) Let p be even, say $p = 2k$. We write $V(K_p) = \{0, 1, 2, \cdots, 2k-1\}$. Consider the graphoidal tree cover $\mathscr{J}_1 = \{T_1, T_2, \cdots, T_k\}$, where each T_i ($i = 1, 2, \cdots, k$) is a spanning tree with edge set defined by

$$
\begin{aligned}
E(T_i) &= \{(i-1, j) : j = i, i+1, \cdots, i+k-1\} \\
&\cup \{(k+i-1, s) : s \equiv j (\mathrm{mod} 2k), j = i+k, i+k+1, \cdots, i+2k-2\}.
\end{aligned}
$$

Now $|\mathscr{J}_1| = k = \frac{p}{2}$. Note that $\Delta(T_i) = k \leq d$ for $i = 1, 2, \cdots, k$ and hence $\gamma_T(d)(K_p) = \lceil \frac{p}{2} \rceil$.

Case (ii) Let p be odd, say $p = 2k+1$. We write $V(K_p) = \{0, 1, 2, \cdots, 2k\}$. Consider the graphoidal tree cover $\mathscr{J}_2 = \{T_1, T_2, \cdots, T_{k+1}\}$ where each T_i ($i = 1, 2, \cdots, k$) is a tree with edge set defined by

$$
\begin{aligned}
E(T_i) &= \{(i-1, j) : j = i, i+1, \cdots, i+k-1\} \\
&\cup \{(k+i-1, s) : s \equiv j (\mathrm{mod} 2k+1), j = i+k, i+k+1, \cdots, i+2k-1\}.
\end{aligned}
$$

$$
E(T_{k+1}) = \{(2k, j) : j = 0, 1, 2, \cdots, k-1\}.
$$

Now $|\mathscr{J}_2| = k = \frac{p}{2}$. Note that the degree of every internal vertex of T_i is either k or $k+1$ and so $\Delta(T_i) \leq d$, $i = 1, 2, \cdots, k+1$. Hence $\gamma_T^{(d)}(K_p) = \lceil \frac{p}{2} \rceil$ if $d \geq \frac{p}{2}$.

Let $d < \frac{p}{2}$. By Corollary 3.2,

$$
\gamma_T^{(d)}(K_p) \geq q+p-pd = \frac{p(p-1)}{2} + p - pd = \frac{p(p-2d+1)}{2}.
$$

Remove the edges from each T_i in $\mathscr{J}_1$ (or $\mathscr{J}_2$) when p is even (odd) so that every internal vertex is of degree d in the new tree T'_i formed by this removal. The new trees so formed together with the removed edges form $\mathscr{J}_3$.

If p is even, then $\mathscr{J}_3$ is constructed from $\mathscr{J}_1$ and

$$|\mathscr{J}_3| = k + q - k(2d-1) = k + \frac{2k(2k-1)}{2} - k(2d-10 = k(2k-2d+1) = \frac{p(p-2d+1)}{2}.$$

If p is odd, then $\mathscr{J}_3$ is constructed from $\mathscr{J}_2$ and

$$|\mathscr{J}_3| = k+1+q-k(2d-1)-d = k+1+\frac{2k(2k+1)}{2}-2kd+k-d = (2k+1)(1+k-d) = \frac{p(p-2d+1)}{2}.$$

Hence $\gamma_T^{(d)}(K_p) = \frac{p(p+1-2d)}{2}$. □

The following examples illustrate the above theorem.

Examples 3.4 Consider K_6. Take $d = 3 = \frac{p}{2}$ and $V(K_6) = \{v_0, v_1, v_2, v_3, v_4, v_5\}$.

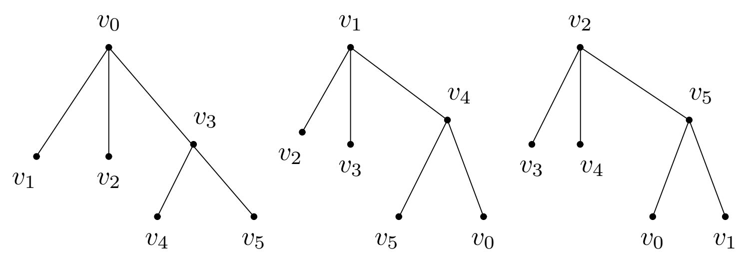

Fig. 1

Whence $\gamma_T^{(3)}(K_6) = 3$. Take $d = 2 < \frac{p}{2}$.

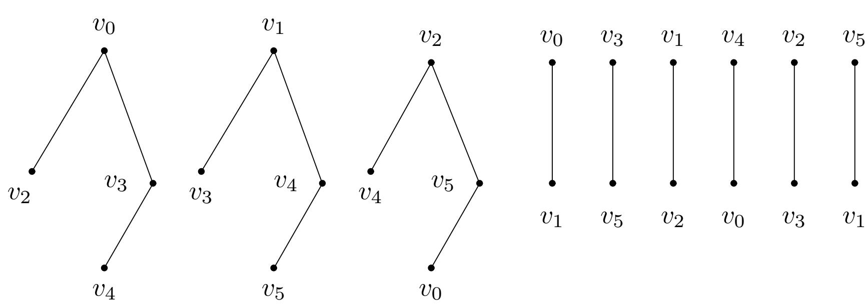

Fig.2

Whence $\gamma_T^{(2)}(K_6) = \frac{6}{2}(6+1-2\times 2) = 9$.

Consider K_7. Take $d = 4 = \lceil \frac{p}{2} \rceil$ and $V(K_7) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$.

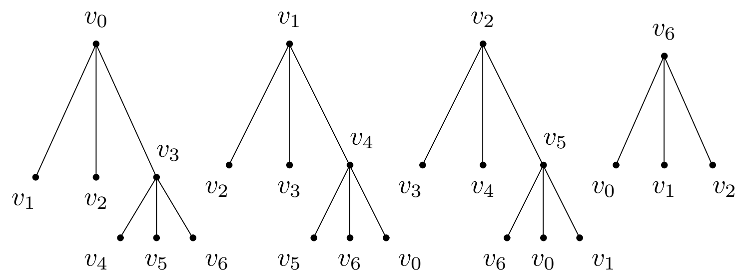

Fig.3

Whence, $\gamma_T^{(4)} = 4 = \lceil \frac{p}{2} \rceil$. Now take $d = 3 < \lceil \frac{p}{2} \rceil$.

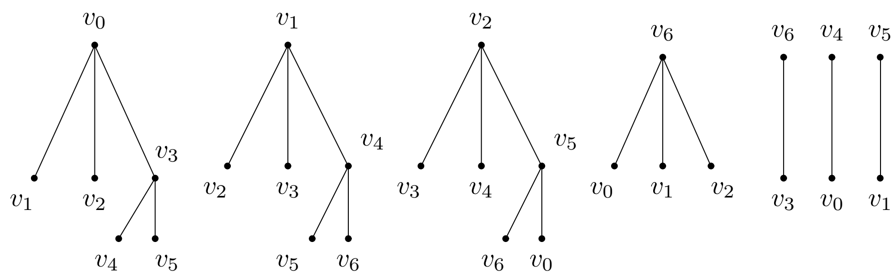

Fig.4

Therefore, $\gamma_T^{(3)}(K_7) = \frac{7}{2}(7 + 1 - 2 \times 3) = 7$.

We now turn to some cases of complete bipartite graph.

Theorem 3.5 *If $n, m \geq 2d$, then $\gamma_T^{(d)}(K_{m,n}) = p + q - pd = mn - (m + n)(d - 1)$.*

Proof By theorem 3.2, $\gamma_T^{(d)}(K_{m,n}) \geq p + q - pd = mn - (m + n)d + m + n$. Consider $G = K_{2d,2d}$. Let $V(G) = X_1 \cup Y_1$, where $X_1 = \{x_1, x_2, \cdots, x_{2d}\}$ and $Y_1 = \{y_1, y_2, \cdots, y_{2d}\}$. Clearly $deg(x_i) = deg(y_j) = 2d$, $1 \leq i, j \leq 2d$. For $1 \leq i \leq d$, we define

$$T_i = \{(x_i, y_j) : 1 \leq j \leq d\}, \quad T_{d+i} = \{(x_{i+d}, y_j) : d + 1 \leq j \leq 2d\}$$

$$T_{2d+i} = \{(y_i, x_j) : d + 1 \leq j \leq 2d\} \text{ and } T_{3d+i} = \{(y_{i+d}, x_j) : 1 \leq j \leq d\}.$$

Clearly, $\mathscr{J} = \{T_1, T_2, \cdots, T_{4d}\}$ is a graphoidal tree d-cover for G. Now consider $K_{m,n}, m, n \geq 2d$. Let $V(K_{m,n}) = X \cup Y$, where $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$. Now for $4d + 1 \leq i \leq 4d + m - 2d = m + 2d$, we define $T_i = \{(x_{i-2d}, y_j) : 1 \leq j \leq d\}$. For $m + 2d + 1 \leq i \leq m + n$, we define $T_i = \{(y_{i-m}, x_j) : 1 \leq j \leq d\}$. Then $\mathscr{J}' = \{T_1, T_2, \cdots, T_{4d}, T_{4d+1}, \cdots, T_{m+2d}, T_{m+2d+1}, \cdots, T_{m+n}\} \cup \{E(G) - [E(T_i) : 1 \leq i \leq m + n]\}$ is a graphoidal tree d-cover for $K_{m,n}$. Hence $|\mathscr{J}'| = p + q - pd$ and so $\gamma_T^{(d)}(K_{m,n}) \leq p + q - pd = mn - (m + n)(d - 1)$ for $m, n \geq 2d$. $\square$

The following example illustrates the above theorem.

Example 3.6 Consider $K_{8,10}$ and take $d = 4$.

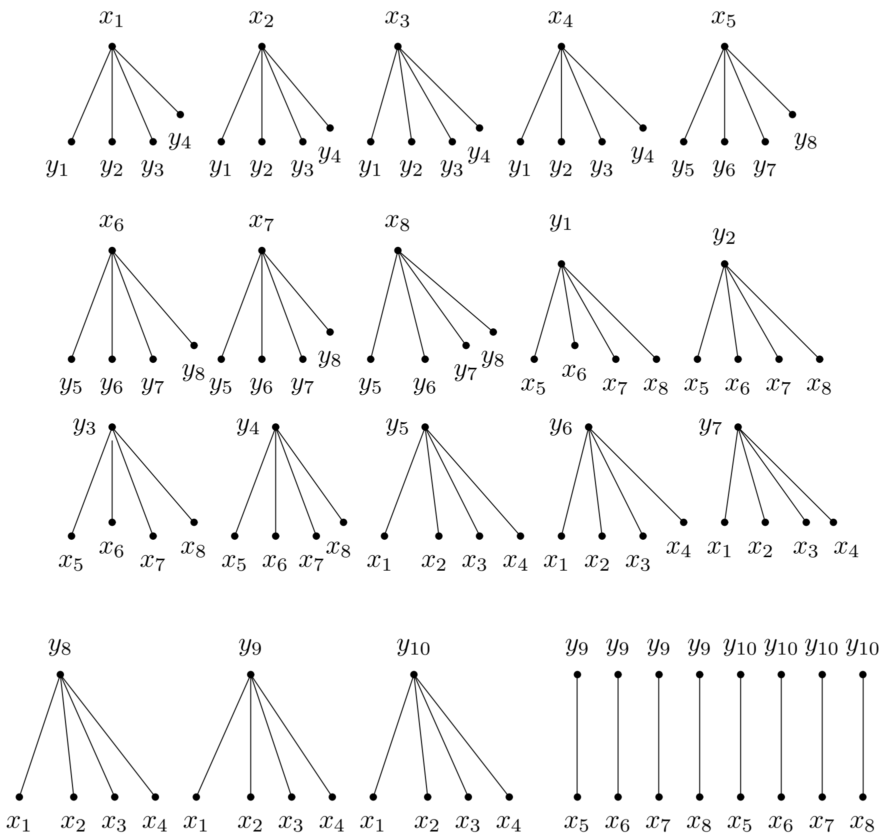

Fig.5

Whence, $\gamma_T^{(4)} = 18 + 80 - 18 \times 4 = 26$.

Theorem 3.7 $\gamma_T^{(d)}(K_{2d-1,2d-1}) = p + q - pd = 2d - 1$.

Proof By Theorem 3.2, $\gamma_T^{(d)}(K_{2d-1,2d-1}) \geq p + q - pd = 2d - 1$. For $1 \leq i \leq d-1$, we define

$$T_i = \{(x_i, y_j) : 1 \leq j \leq d\} \cup \{(y_i, x_{d+j}) : 1 \leq j \leq d-1\} \cup \{(x_{d+i}, y_{d+j}) : 1 \leq j \leq d-1\}.$$

Let $T_d = \{(x_d, y_j) : 1 \leq j \leq d\} \cup \{(y_d, x_{d+j}) : 1 \leq j \leq d-1\}$. For $d+1 \leq i \leq 2d-1$, we define $T_i = \{(y_i, x_j) : 1 \leq j \leq d\}$. Clearly $\mathscr{J} = \{T_1, T_2, \cdots, T_{2d-1}\}$ is a graphoidal tree d-cover of G and so

$$\gamma_T^{(d)}(G) \leq 2d - 1 = (2d-1)(2d-1-2(d-1)) = q + p - pd.$$

□

The following example illustrates the above theorem.

Example 3.8 Consider $K_{9,9}$ and $d = 5$.

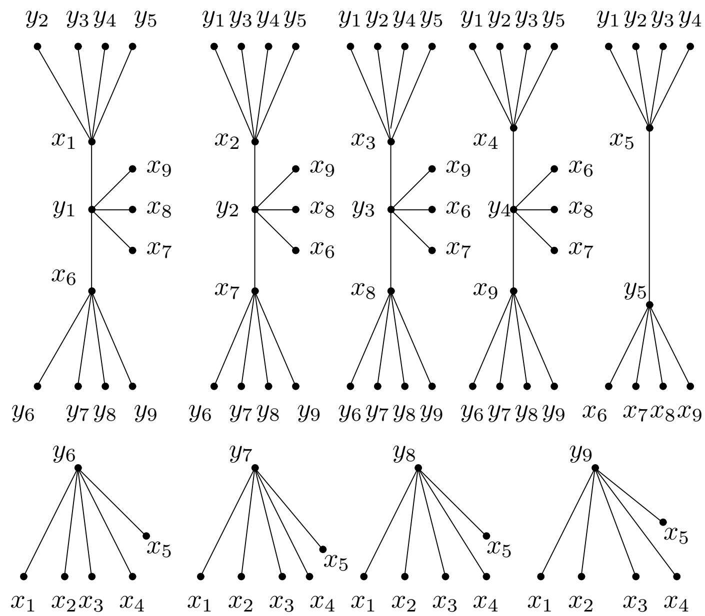

Fig.6

Thereafter, $\gamma_T^{(5)}(K_{9,9}) = 81 + 18 - 90 = 9$.

Lemma 3.9 $\gamma_T^{(d)}(K_{3r,3r}) \leq 2r$, *where* $d \geq 2r$ *and* $r \geq> 1$.

Proof Let $V(K_{3r,3r}) = X \cup Y$, where $X = \{x_1, x_2, \cdots, x_{3r}\}$ and $Y = \{y_1, y_2, \cdots, y_{3r}\}$.

Case (i) r is even.

For $1 \leq s \leq r$, we define

$$\begin{aligned} T_s &= \{(x_s, y_{s+i}) : 0 \leq i \leq r-1\} \cup \{(x_s, y_{2r+s})\} \cup \{(x_{r+s}, y_{2r+s})\} \cup \{(y_{2r+s}, x_{2r+s})\} \\ &\cup \{(x_i, y_{2r+s}) : 1 \leq i \leq r, i \neq s\} \cup \{(x_{r+s}, y_i) : r+s \leq i \leq 3r, i \neq 2r+s\} \\ &\cup \{(x_{r+s}, y_i) : 1 \leq i \leq s-1, s \neq 1\} \end{aligned}$$

and

$$\begin{aligned} T_{r+s} &= \{(y_s, x_{s+i}) : 1 \leq i \leq r\} \cup \{(y_s, x_{2r+s})\} \cup \{(y_{r+s}, x_{2r+s})\} \\ &\cup \{(y_i, x_{2r+s}) : 1 \leq i \leq r, i \neq s, 2r+1 \leq i \leq 3r, i \neq 2r+s\} \\ &\cup \{(y_{r+s}, x_i) : r+s+1 \leq i \leq 3r, 1 \leq i \leq s, i \neq 2r+s\}. \end{aligned}$$

Then $\mathscr{J}_1 = \{T_1, T_2, \cdots, T_{2r}\}$ is a graphoidal tree d-cover for $K_{3r,3r}$, $\Delta(T_i) \leq 2r$ and $d \geq 2r$. So we have, $\gamma_T^{(d)}({}_{K3r,3r}) \leq 2r$.

Case (ii) r is odd.

For $1 \leq s \leq r$, we define

$$\begin{aligned} T_s &= \{(x_s, y_{s+i}) : 0 \leq i \leq 2r-1\} \cup \{(y_{r+s}, x_i) : r+1 \leq i \leq 3r, i \neq r+s\} \\ &\cup \; \{(x_{2r+s}, y_i) : 2r+s \leq i \leq 3r\} \cup \{(x_{2r+s}, y_i) : 1 \leq i \leq s-1, s \neq 1\} \end{aligned}$$

$$\begin{aligned} T_{r+s} &= \{(y_s, x_{s+i}) : 1 \leq i \leq 2r\} \cup \{(x_{r+s}, y_i) : 2r+1 \leq i \leq 3r; i = r+s\} \\ &\cup \; \{(y_{2r+s}, x_i) : 2r+s+1 \leq i \leq 3r, s \neq r\} \cup \{(y_{2r+s}, x_i) : 1 \leq i \leq s\}. \end{aligned}$$

Clearly $\Delta(T_i) \leq 2r$ for each i. In this case also $\mathscr{J}_2 = \{T_1, T_2, \cdots, T_{2r}\}$ is a graphoidal tree d-cover for $K_{3r,3r}$ and so $\gamma_T^{(d)}(K_{3r,3r}) \leq 2r$ when r is odd. $\square$

The following example illustrates the above lemma for $r = 2, 3$. Consider $K_{6,6}$ and $K_{9,9}$.

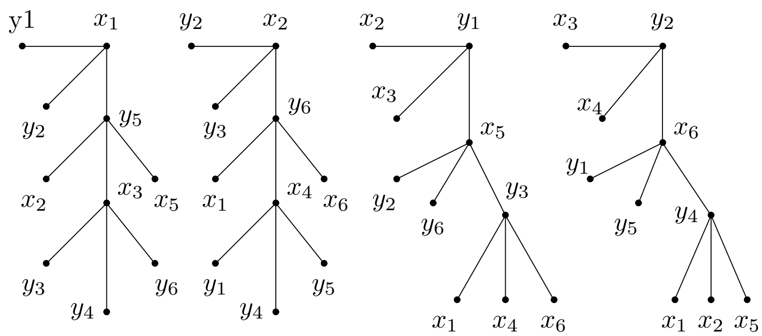

Fig.7

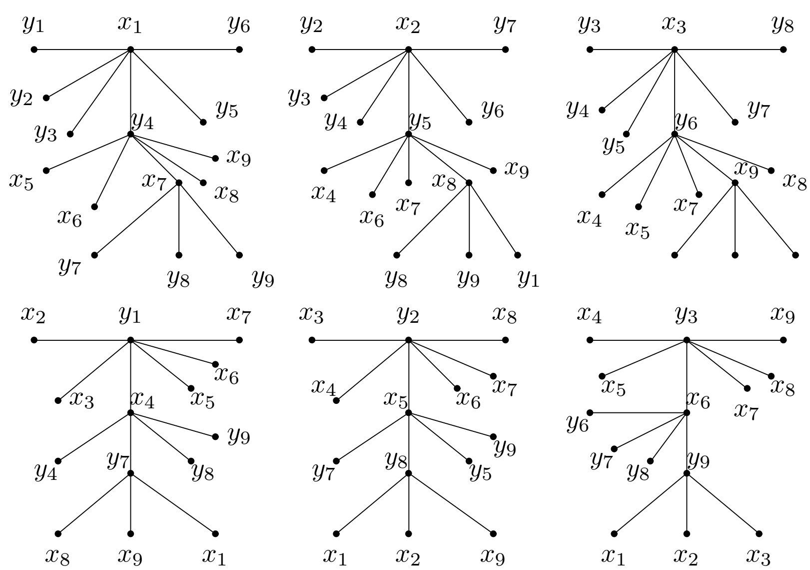

Fig.8

Theorem 3.10 *$\gamma_T^{(d)}(K_{n,n}) = \lceil \frac{2n}{3} \rceil$ for $d \geq \lceil \frac{2n}{3} \rceil$ and $n > 3$.*

Proof By Theorem 2.2, $\lceil \frac{2n}{3} \rceil = \gamma_T(K_{n,n})$ and $\gamma_T(K_{n,n}) \leq \gamma_T^{(d)}(K_{n,n})$, it follows that $\gamma_T^{(d)}(K_{n,n}) \geq \lceil \frac{2n}{3} \rceil$ for any n. Hence the result is true for $n \equiv 0(\mathrm{mod}3)$. Let $n \equiv 1(\mathrm{mod}3)$ so that $n = 3r+1$ for some r. Let $\mathscr{J}_1 = \{T_1', T_2', \cdots, T_{2r}'\}$ be a minimum graphoidal tree d-cover for $K_{3r,3r}$ as in Lemma 3.9. For $1 \leq i \leq r$, we define

$T_i = T_i' \cup \{(x_i, y_{3r+1})\}$,

$T_{r+i} = T_{r+i}' \cup \{(y_i, x_{3r+1})\}$ and

$T_{2r+1} = \{(x_{3r+1}, y_{r+i}) : 1 \leq i \leq 2r+1\} \cup \{(y_{3r+1}, x_{r+i}) : 1 \leq i \leq 2r\}$.

Clearly $\mathscr{J}_2 = \{T_1, T_2, \cdots, T_{2r+1}\}$ is a graphoidal tree d-cover for $K_{3r+1,3r+1}$, as $\Delta(T_i) \leq 2r+1 = \lceil \frac{2n}{3} \rceil \leq d$ for each i. Hence $\gamma_T^{(d)}(K_{n,n}) = \gamma_T^{(d)}(K_{3r+1,3r+1}) \leq 2r+1 = \lceil \frac{2n}{3} \rceil$.

Let $n \equiv 2(\mathrm{mod}3)$ and $n = 3r+2$ for some r. Let $\mathscr{J}_3$ be a minimum graphoidal tree d-cover for $K_{3r+1,3r+1}$ as in the previous case. Let $\mathscr{J}_3 = \{T_1, T_2, \cdots, T_{2r+1}\}$. For $1 \leq i \leq r$, we define

$T_i' = T_i \cup \{(x_i, y_{3r+2})\}$,

$T_{r+i}' = T_{r+i} \cup \{(y_i, x_{3r+2})\}$,

$T_{2r+1}' = T_{2r+1}$,

$T_{2r+2}' = \{(x_{3r+2}, x_{r+i}) : 1 \leq i \leq 2r+2\} \cup \{(y_{3r+2}, x_{r+i}) : 1 \leq i \leq 2r+1\}$.

Clearly, $\mathscr{J}_4 = \{T_1', T_2', \cdots, T_{2r+2}'\}$ is a graphoidal tree d-cover for $K_{3r+2,3r+2}$, as $\Delta(T_i') \leq 2r+2 = \lceil \frac{2n}{3} \rceil \leq d$ for each i. Hence $\gamma_T^{(d)}(K_{n,n}) = \gamma_T^{(d)}(K_{3r+2,3r+2}) \leq 2r+2 = \lceil \frac{2n}{3} \rceil$. Therefore, $\gamma_T^{(d)}(K_{n,n}) = \lceil \frac{2n}{3} \rceil$ for every n. □

Now we turn to the case of trees.

Theorem 3.11 *Let G be a tree and let $U = \{v \in V(G) : \ deg(v) - d > 0\}$. Then $\gamma_T^{(d)}(G) = \sum\limits_{v \in V(G)} \chi_U(v)(deg(v) - d) + 1$, where $d \geq 2$ and $\chi_U(v)$ is the characteristic function of U.*

Proof The proof is by induction on the number of vertices m whose degrees are greater than d. If $m = 0$, then $\mathscr{J} = G$ is clearly a graphoidal tree d-cover. Hence the result is true in this case and $\gamma_T^{(d)}(G) = 1$. Let $m > 0$. Let $u \in V(G)$ with $deg_G(u) = d + s$ ($s > 0$). Now decompose G into $s+1$ trees $G_1, G_2, \cdots, G_s, G_{s+1}$ such that $deg_{G_i}(u) = 1$ for $1 \leq i \leq s$, $deg_{G_{s+1}}(u) = d$. By induction hypothesis,

$$\gamma_T^{(d)}(G_i) = \sum_{deg_{G_i}(v) > d} (deg_{G_i} - d) + 1 = k_i, \quad 1 \leq i \leq s+1.$$

Now $\mathscr{J}_i$ is the minimum graphoidal tree d-cover of G_i and $|\mathscr{J}_i| = k_i$ for $1 \leq i \leq s+1$. Let $\mathscr{J} = \mathscr{J}_1 \cup \mathscr{J}_2 \cup \cdots \cup \mathscr{J}_{s+1}$.

Clearly $\mathscr{J}$ is a graphoidal tree d-cover of G. By our choice of u, u is internal in only one tree T of $\mathscr{J}$. More over, $deg_T(u) = d$ and $deg_{G_i}(v) = deg_G(v)$ for $v \neq u$ and $v \in V(G_i)$ for $1 \leq i \leq s+1$. Therefore,

$$\begin{aligned}
\gamma_T^{(d)} &\leq |\mathscr{J}| = \sum_{i=1}^{s+1} k_i = \sum_{i=1}^{s+1}\left[\sum_{deg_{G_i}(v)>d}(deg_{G_i}(v)-d)+1\right] \\
&= \sum_{i=1}^{s+1}\left[\sum_{deg_{G_i}(v)>d}(deg_{G_i}(v)-d)\right]+s+1 = \sum_{deg_G(v)>d,v\neq u}(deg_G(v)-d)+s+1 \\
&= \sum_{deg_G(v)>d,v\neq u}(deg_G(v)-d)+(deg_G(u)-d)+1 = \sum_{deg_G(v)>d}(deg_G(v)-d)+1 \\
&= \sum_{v\in V(G)}\chi_U(v)(deg_G(v)-d)+1.
\end{aligned}$$

For each $v \in V(G)$ and $deg_G(v) > d$ there are at least $deg_G(v) - d + 1$ subtrees of G in any graphoidal tree d-cover of G and so $\gamma_T^{(d)}(G) \geq \sum\limits_{deg_G(v)>d}(deg_G(v)-d)+1$. Hence $\gamma_T^{(d)}(G) = \sum\limits_{v\in V(G)}\chi_U(v)(deg_G(v)-d)+1$. □

Corollary 3.12 *Let G be a tree in which degree of every vertex is either greater than or equal to d or equal to one. Then $\gamma_T^{(d)}(G) = m(d-1)-p(d-2)-1$, where m is the number of vertices of degree 1 and $d \geq 2$.*

Proof Since all the vertices of G other than pendant vertices have degree d we have,

$$\begin{aligned}
\gamma_T^{(d)} &= \sum_{v\in V(G)}\chi_U(v)(deg_G(v)-d)+1 = \sum_{v\in V(G)}\chi_U(v)(deg_G(v)-d)+md-m+1 \\
&= 2q-dp+md-m+1 = 2p-2-dp+md-m+1 \quad (\text{as } q=p-1) \\
&= m(d-1)-p(d-2)-1.
\end{aligned}$$

□

Recall that $n_d = \min\limits_{\mathscr{J}\in\mathscr{G}_d} n_{\mathscr{J}}$ and $n = \min\limits_{\mathscr{J}\in\mathscr{G}} n_{\mathscr{J}}$, where $\mathscr{G}_d$ is the collection of all graphoidal tree d-covers of G, $\mathscr{G}$ is the collection of all graphoidal tree covers of G and $n_{\mathscr{J}}$ is the number of vertices which are not internal vertices of any tree in $\mathscr{J}$. Clearly $n_d = n$ if $d \geq \Delta$. Now we prove this for any $d \geq 2$.

Lemma 3.13 *For any graph G, $n_d = n$ for any integer $d \geq 2$.*

Proof Since every graphoidal tree d-cover is also a graphoidal tree cover for G, we have $n \leq n_d$. Let $\mathscr{J} = \{T_1, T_2, \cdots, T_m\}$ be any graphoidal tree cover of G. Let Ψ_i be a minimum graphoidal tree d - cover of T_i $(i = 1, 2, \cdots, m)$. Let $\Psi = \bigcup\limits_{i=1}^{m}\Psi_i$. Clearly Ψ is a graphoidal tree d-cover of G. Let n_Ψ be the number of vertices which are not internal in any tree of Ψ. Clearly $n_\Psi = n_{\mathscr{J}}$. Therefore, $n_d \leq n_\Psi = n_{\mathscr{J}}$ for $\mathscr{J} \in \mathscr{G}$, where $\mathscr{G}$ is the collection of graphoidal tree covers of G and so $n_d \leq n$. Hence $n = n_d$. □

We have the following result for graphoidal path cover. This theorem is proved by S.

Arumugam and J. Suresh Suseela in [5]. We prove this, by deriving a minimum graphoidal path cover from a graphoidal tree cover of G.

Theorem 3.14 $\gamma_T^{(2)}(G) = q - p + n_2$.

Proof From Theorem 3.1 it follows that $\gamma_T^{(2)}(G) \geq q - p + n_2$. Let $\mathscr{J}$ be any graphoidal tree cover of G and $\mathscr{J} = \{T_1, T_2, \cdots, T_k\}$. Let Ψ_i be a minimum graphoidal tree d-cover of T_i $(i = 1, 2, \cdots, k)$. Let m_i be the number of vertices of degree 1 in T_i $(i = 1, 2, \cdots, k)$. Then by Theorem 3.12 it follows that $\gamma_T^{(2)}(T_i) = m_i - 1$ for all $i = 1, 2, \cdots, k$. Consider the graphoidal tree 2-cover $\Psi_{\mathscr{J}} = \bigcup\limits_{i=1}^{k} \Psi_i$ of G. Now

$$\begin{aligned} |\Psi_{\mathscr{J}}| = \sum_{i=1}^{k} |\Psi_i| &= \sum_{i=1}^{k}(m_i - 1) = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k} q_i - \sum_{i=1}^{k} p_i \\ &= q - \sum_{i=1}^{k} p_i + \sum_{i=1}^{k} m_i. \end{aligned}$$

Notice that

$$\begin{aligned} \sum_{i=1}^{k} p_i &= \sum_{i=1}^{k} (\text{numbers of internal vertices and pendant vertices of } T_i) \\ &= p - n_{\mathscr{J}} + \sum_{i=1}^{k} m_i. \end{aligned}$$

Therefore, $|\Psi_{\mathscr{J}}| = q - p + n$. Choose a graphoidal tree cover $\mathscr{J}$ of G such that $n_{\mathscr{J}} = n$. Then for the corresponding $\Psi_{\mathscr{J}}$ we have $|\Psi_{\mathscr{J}}| = q - p + n = q - p + n_2$, as $n_2 = n$ by Lemma 3.13. □

Corollary 3.15 *If every vertex is an internal vertex of a graphoidal tree cover, then* $\gamma_T^{(2)}(G) = q - p$.

Proof Clearly $n = 0$ by definition. By Lemma 3.13, $n_2 = n$. So we have $n_2 = 0$. □

J. Suresh Suseela and S. Arumugam proved the following result in [5]. However, we prove the result using graphoidal tree cover.

Theorem 3.16 *Let G be a unicyclic graph with r vertices of degree 1. Let C be the unique cycle of G and let m denote the number of vertices of degree greater than 2 on C. Then*

$$\gamma_T^{(2)}(G) = \begin{cases} 2 & if\ m = 0, \\ r + 1 & m = 1,\ deg(v) \geq 3 \ where\ v\ is\ the\ unique\ vertex\ of\ degree > 2\ on\ C, \\ r & oterwise. \end{cases}$$

Proof By Lemma 3.13 and Theorem 3.14, we have $\gamma_T^{(2)}(G) = q-p+n$. We have $q(G) = p(G)$ for unicyclic graph. So we have $\gamma_T^{(2)}(G) = n$. If $m = 0$, then clearly $\gamma_T^{(2)}(G) = 2$. Let $m = 1$ and let v be the unique vertex of degree > 2 on C. Let $e = vw$ be an edge on C. Clearly $\mathscr{J} = G - e, e$ is a minimum graphoidal tree cover for G and so $n \leq r + 1$. Since there is a vertex of C which is not internal in a tree of a graphoidal tree cover, we have $n = r+1$. When $m = 1$, $\gamma_T^{(2)}(G) = r+1$. Let $m \geq 2$. Let v and w be vertices of degree greater than 2 on C such that all vertices in a (v, w) - section of C other than v and w have degree 2. Let P denote this (v, w)-section. If P has length 1. Then $P = (v, w)$. Clearly $\mathscr{J} = G - P, P$ is a graphoidal tree cover of G. Also $n = r$ and so $\gamma_T^{(2)}(G) = r$ when $m \geq 2$. Hence we get the theorem. □

Theorem 3.17 *Let G be a graph such that $\gamma_T^{(G)} \leq \delta(G) - d + 1$ ($\delta(G) > d \geq 2$). Then $\gamma_T^{(d)}(G) = q - p(d-1)$.*

Proof By Theorem 3.2, $\gamma_T^{(d)}(G) \geq q-p(d-1)$. Let $\mathscr{J}$ be a minimum graphoidal tree cover of G. Since $\delta > \gamma_T(G)$, every vertex is an internal vertex of a tree in a graphoidal tree cover $\mathscr{J}$. Moreover, since $\delta \geq d+\delta_T(G)-1$ the degree of each internal vertex of a tree in $\mathscr{J}$ is $\geq d$. Let Ψ_i be a minimum graphoidal tree d-cover of T_i ($i = 1, 2, \cdots, k$). Let m_i be the number of vertices of degree 1 in T_i ($i = 1, 2, \cdots, k$). Then by Corollary 3.12, for $i = 1, 2, \cdots, k$ we have

$$\gamma_T^{(2)}(T_i) = -p_i(d-2) + m_i(d-1) - 1.$$

Consider the graphoidal tree d-cover $\Psi_T = \bigcup\limits_{i=1}^{k} \Psi_i$ of G.

$$\begin{aligned}
|\Psi_T| &= |\bigcup_{i=1}^{k} \Psi_i| = \sum_{i=1}^{k}(m_i(d-1) - p_i(d-2) - 1) \\
&= \sum_{i=1}^{k}(m_i(d-1) - p_i(d-2) + q_i - p_i) \\
&= \sum_{i=1}^{k}[(m_i - p_i)(d-1) + q_i] \\
&= (d-1)\sum_{i=1}^{k}(m_i - p_i) + \sum_{i=1}^{k} q_i \\
&= (d-1)\sum_{i=1}^{k}(m_i - p_i) + q.
\end{aligned}$$

Notice that

$$\begin{aligned}
\sum_{i=1}^{k} p_i &= \sum_{i=1}^{k}(\text{numbers of internal vertices and pendant vertices of } T_i) \\
&= p + \sum_{i=1}^{k} m_i.
\end{aligned}$$

Therefore, $|\Psi_T| = -(d-1)+q$. In other words, $\gamma_T^{(d)}(G) \leq q - p(d-1)$. Hence, $\gamma_T^{(d)}(G) = q - p(d-1)$ □

Corollary 3.18 *Let G be a graph such that $\delta(G) = \lceil \frac{p}{2} \rceil + k$ where $k \geq 1$. Then $\gamma_T^{(d)}(G) = q - p(d-1)$ for $d \leq k+1$.*

Proof $\delta(G) - d + 1 = \lceil \frac{p}{2} \rceil + k - d + 1 \geq \lceil \frac{p}{2} \rceil \geq \gamma_T(G)$ by Theorem 2.5. Applying Theorem 3.17, $\gamma_T^{(d)}(G) = q - p(d-1)$. □

Corollary 3.19 *Let G be an r-regular graph, where $r > \lceil \frac{p}{2} \rceil$. Then $\gamma_T^{(d)}(G) = q - p(d-1)$ for $d \leq r + 1 - \lceil \frac{p}{2} \rceil$.*

Proof Here $\delta(G) = r$ and so the result follows from Corollary 3.18. □

Corollary 3.20 $\gamma_T^{(d)}(K_{m,n}) = q - p(d-1)$, *where* $2 \leq d \leq \frac{2m-n}{3}$ *and* $6 \leq m \leq n \leq 2m-6$.

Proof Consider

$$\begin{aligned}\delta(G) - d + 1 &\geq m - \frac{2m-n}{3} + 1 = \frac{3m-2m+n}{3} + 1 \\ &= \frac{m+n}{3} + 1 \geq \lceil \frac{m+n}{3} \rceil = \gamma_T(K_{m,n}).\end{aligned}$$

Hence by Corollary 3.18, $\gamma_T^{(d)}(K_{m,n}) = q - p(d-1)$. □

Theorem 3.21 $\gamma_T^{(d)}(C_m \times C_n) = 3$ *for* $d \geq 4$ *and* $\gamma_T^{(2)}(C_m \times C_n) = q - p$.

Proof For $d \geq \Delta(G) = 4$, $\gamma_T^{(d)}(C_m \times C_n) = \gamma_T(C_m \times C_n) = 3$ by Theorem 2.14. Since $\delta(C_m \times C_n) = 4$ and $\gamma_T(C_m \times C_n) = 3$, we have $\gamma_T(C_m \times C_n) = \delta(G) - d + 1$ when $d = 2$. Applying Theorem 3.17, $\gamma_T^{(2)}(C_m \times C_n) = q - p$. □

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Comparing the Number of Acyclic and Totally Cyclic Orientations with That of Spanning Trees of a Graph

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Abstract: A Smarandache k-orientation of G for an integer $k \geq 0$ is such an orientation on G with exactly k oriented cycles. If $k = 0$, then it is the common acyclic orientation. In C. Merino and D.J.A.Welsh, *Forests, colourings and acyclic orientations of the square lattice*, Annals of Combinatorics, 3 (1999), pp. 417 C 429, the following conjecture appears: *If G is a 2-connected graph with no loops, then either the number of acyclic orientations or the number of the totally cyclic orientations of G is bigger than the number of spanning trees of G.* In this paper we examine this conjecture for threshold graphs, which includes complete graphs, and complete bipartite graphs. Also we show the results of our computational search for a counterexample to the conjecture.

Key Words: Smarandache k-tree, Smarandache k-orientation, Tutte polynomial, spanning tree, acyclic orientation.

AMS(2000): 05A15, 05A20, 05C17.

§1. Preliminaries

The graph terminology that we use is standard and we follow Diestel's notation, see [7]. We consider throughout labelled connected graphs which may have loops and multi-edges. For a graph G we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. We denote by $G \setminus e$ the deletion of e from G and by G/e the contraction of e in G.

A Smarandache k-tree $\mathscr{T}^k$ for an integer $k \geq 0$ is a connected spanning subgraph of a connected graph G with exactly k cycles. Particularly, if $k = 0$, i.e., $\mathscr{T}^0$ is the commonly spanning tree which is a connected acyclic subgraph of G. A Smarandache k-orientation of G is such an orientation on G with exactly k oriented cycles for an integer $k \geq 0$. Particularly, a

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Smarandache 0-orientation is the common acyclic orientation on G which contains no oriented cycle, and it is totally cyclic if every arc is part of an oriented cycle. When G is connected, the totally cyclic orientations correspond to the orientations which make G strongly connected. The number of spanning trees, acyclic and totally cyclic orientations of a connected G are denoted by $\tau(G)$, $\alpha(G)$ and $\alpha^*(G)$, respectively.

1.1 Tutte Polynomial

As all the invariants mentioned here are evaluations of the Tutte polynomial, let us define it and rephrase the conjecture in this setting.

The Tutte polynomial $T(G;x,y)$ of a graph G in variables x,y is $T(G;x,y)=1$ if G has no edges, otherwise for any $e \in E(G)$:

$\mathbf{R}_1$ $T(G;x,y) = xT(G/e;x,y)$, whenever e is a bridge;

$\mathbf{R}_2$ $T(G;x,y) = yT(G \setminus e;x,y)$, whenever e is a loop;

$\mathbf{R}_3$ $T(G;x,y) = T(G \setminus e;x,y) + T(G/e;x,y)$, otherwise.

In other words, T may be calculated recursively by choosing the edges in any order and repeatedly using the relations $R_1C\ R_3$ to evaluate T. The resulting polynomial T is well defined in the sense that it is independent of the order in which the edges are chosen, see [3].

When evaluating the Tutte polynomial along different curves and points we get several interesting invariants of graphs. Among them we have the chromatic and flow polynomials of a graph; the all terminal reliability probability of a network; the partition function of a Q-state Potts model. But here we are interested in the evaluations at the three points $T(G;1,1)=\tau(G)$, $T(G;2,0)=\alpha(G)$ and $T(G;0,2)=\alpha^*(G)$. Further details of many of the invariants given by evaluations of the Tutte polynomial can be found in [21] and [5].

§2. Introduction

In 1999, Merino and D.J.A. Welsh were working in the asymptotic behavior of the number of spanning trees and acyclic orientations of the square lattice Ln. They foresaw that

$$\lim_{n\to\infty}(\tau(L_n))^{\frac{1}{n^2}} < \lim_{n\to\infty}(\alpha(L_n))^{\frac{1}{n^2}}.$$

(the result was later proved in [6]). So, for some big N and all $n \geq N$, $\tau(L_n) < \alpha(L_n)$. They also checked this for some small values of n. No proof of the inequality $\tau(L_n) < \alpha(L_n)$ for all n has been given yet. It is obvious that $\alpha(G) \geq \tau(G)$ is not true for general graphs, like for example the complete graphs K_n for $n \geq 5$, but it is true for the cycles C_n for all n. By considering the dual concept (more than the dual matroids, as it was important to stay in the class of graphic matroids), namely the quantity α^*, they convinced themselves that $\alpha^*(K_n) \geq \tau(K_n)$ for $n \geq 5$ was true but did not prove it formally. This small piece of evidence was enough to pose the following

Conjecture 2.1 *Let G be a 2-connected graph with no loops, then*

$$max\{\alpha(G), \alpha^*(G)\} \geq \tau(G).$$

The conjecture can be stated in terms of the Tutte polynomial: *let G be a 2-connected graph with no loops, then*

$$max\{T(G; 2, 0), T(G; 0, 2)\} \geq T(G; 1, 1).$$

First let us consider why the conditions are necessary. Even though the conjecture is true for trees, since $\tau(T_n) = 1 < 2^n = \alpha(T_n)$, where T_n is a tree with n edges, in general we need the condition of G being 2-connected. Take two graphs G_1 and G_2 with the properties that $0 < \alpha(G_1) < \frac{1}{\sqrt{2}}\tau(G_1)$, like K_6, and $0 < \alpha^*(G_2) < \frac{1}{\sqrt{2}}\tau(G_2)$, like C_5, to have a couple of counterexamples. One, by taking two copies of G_1 and joining them by an edge; this new graph G_3 has a bridge and it is such that $\alpha(G_3) = 2\alpha^2(G_1) < \tau^2(G_1) = \tau(G_1)$ and $\alpha^*(G_3) = 0$. The other one, by attaching a loop to G_2 to obtain G_4; in this case $\alpha^*(G_4) = 2\alpha(G_2) < \tau(G_2) = \tau(G_4)$ and $\alpha(G_4) = 0$. Also, the conjecture can be strengthened in several ways.

Conjecture 2.2 *Let G be a 2-connected graph with no loops, then*

$$T(G; 2, 0)T(G; 0, 2) \geq T^2(G; 1, 1).$$

If true, this conjecture would imply Conjecture 2.1. We denote the quantity $T(G; 2, 0)T(G; 0, 2)$ by $\gamma(G)$. We use this conjecture for the class of threshold graphs. Another such strengthened is the following:

Conjecture 2.3 *Let G be a 2-connected graph with no loops, then*

$$T(G; 2, 0) + T(G; 0, 2) \geq 2T(G; 1, 1).$$

The interest in the conjecture goes beyond being an interesting combinatorial problem. As seen by Conjecture 2.3 the problem is a first step towards checking the convexity of the Tutte polynomial along the line from $(2, 0)$ to $(0, 2)$. Here is a rather more daring strengthening of the conjecture.

Conjecture 2.4 *Let G be a 2-connected simple graph with no loops, then $T(G; x, y)$ is convex along the the line from $(2, 0)$ to $(0, 2)$.*

Our interest in the conjecture is in trying to understand the structure of the Tutte polynomials, in particular, we believe there is a combinatorial structure in the graph G corresponding to the evaluation along the segment $x = 1 + t$, $y = 1 - t$ for $-1 \leq t \leq 1$.

This article tries to provide further evidence for the truth of Conjecture 2.1 to make it more plausible. The rest of the paper contains 3 more sections. In Section 3 we provide different classes of graphs which satisfy the conjecture. Then, we explain some experimental results that validate the conjecture for graphs with a small number of vertices. The final section contains our conclusions.

§3. Theoretical evidence of the conjecture

Now, let us give some infinite families of graphs for which the conjecture is true. Evidently for the examples of cycles C_n we have that $\alpha^*(C_n) = 2^n - 2 \geq n = \tau(C_n)$ for $n \geq 2$. By duality the conjecture is also true for the graphs P_2^n with 2 vertices and n edges in parallel.

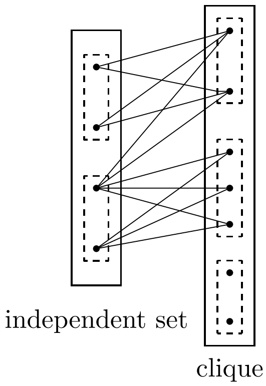

Fig.1 A threshold graph with $\omega(G) = 7$

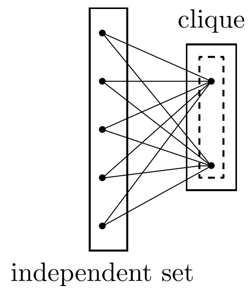

Fig.2 The case of Lemma 3.3, $K_{2,m}^+$

3.1 Threshold graphs

Threshold graphs are an important class of perfect graphs, see [16], and they can be considered a generalization of complete graphs with which they share many properties. A threshold graph is a simple graph $G = (V, E)$ with $V = \{1, \cdots, n\}$ such that

(1) for $\omega(G) \geq 2$, the clique number of G, we have that the first $\omega(G)$ vertices of G induce a $K_{\omega(G)}$ while the last $n - \omega(G)$ vertices form an independent set;
(2) for $1 \leq i \leq j \leq n$, we have that $N(j) \subset N(i)$, where $N(i)$ denotes the neighborhood of i.

Thus, a threshold graph is a complete graph $K_{\omega(G)}$ to which we add $n - \omega(G)$ simplicial vertices, one by one, preserving the nested neighborhood. Threshold graphs are a kind of chordal graph. Observe that, if G is not the complete graph, $N(\omega(G) + 1) \subset N(\omega(G))$.

We assume that $d_1 \geq d_2 \geq \cdots \geq d_n$, where d_i is the degree of vertex i. We are concerned just with $d_n \geq 2$ which is the same as requiring the graph to be 2-connected. When $\omega(G) = n$, we have a complete graph.

One of the nice properties of threshold graphs is that the number of spanning trees and the chromatic polynomial can be computed easily. In fact, for G, a connected threshold graph, we have that

$$\tau(G) = \prod_{i=2}^{\omega(G)-1} (d_i + 1) \prod_{i=\omega(G)+1}^{n} d_i. \qquad (3-1)$$

The formula follows directly from a result due to R. Merris in [14]. Observe that when $\omega(G) = n$, the formula is Cayleys formula for the number of spanning trees of K_n. The chromatic polynomial can be computed by using a classical result of Read on quasi-separations, see [17], we have that

$$\chi(G;x) = \prod_{i=1}^{\omega(G)} (x-i+1) \prod_{i=\omega(G)+1}^{n} (x-d_i).$$

Now, applying the famous result in [19] that relates the number of acyclic orientations and the chromatic polynomial, namely $\alpha(G) = |\chi(G;-1)|$, we get that

$$\alpha(G) = \omega(G)! \prod_{i=\omega(G)+1}^{n} (d_i+1). \qquad (3-2)$$

We start by giving some useful lemmas.

Lemma 3.1 *For $x \geq 2$,*

$$ln(x+1) - ln(x) < \frac{1}{x+\frac{1}{3}}.$$

Lemma 3.2 *If G is a 2-connected graph with a vertex v of degree d, then $(2^d-2)\alpha^*(G-v) \leq \alpha^*(G)$.*

Proof If $\alpha^*(G-v) = 0$, the result is trivial. Otherwise, choose any totally cyclic orientation θ_{G-v} of $G-v$. Now, choose θ_v as any of the 2^d-2 orientations of the edges adjacent to v that do not make it a source or a sink. We extend θ_{G-v} to an orientation θ_G of G by oriented the edges on v according to θ_v. We prove that this orientation is totally cyclic. Any arc $u \to w$ with $u \neq v \neq w$ is in a oriented cycle of θ_{G-v}. To an arc $\overrightarrow{a} = v \to u$ corresponds an arc $\overrightarrow{b} = w \to v$ by the choice of θ_v. The orientation θ_{G-v} makes $G-v$ strongly connected, so there is a directed path from u to w; that together with $w \to v \to u$ form a directed cycle. Thus, both $\overrightarrow{a}$ and $\overrightarrow{b}$ are in an oriented cycle. $\square$

We prove that threshold graphs satisfy Conjecture 2.2 in three stages.

Lemma 3.3 *If G is a threshold graph with degree sequence $(d_1, \cdots, d_n)$, $d_n = \cdots = d_3 = 2$, and $\omega(G) = 3$, then $\gamma(G) \geq \tau^2(G)$.*

Proof The graph G is $K_{2,m}$ plus the edge e_0 joining the two vertices of degree m, where $m = n-2$. We denote this graph by $K_{2,m}^+$. By the deletion and contraction relation $\mathbf{R}_3$ applied to e_0, we get that the Tutte polynomial of G is the sum of the Tutte polynomials of the graphs St_m^2 and $K_{2,m}$, where St_m^2 is the graph obtained from a star with m edges by replacing every edge by a 2-cycle. The first graph has Tutte polynomial $(x+y)^m$. The Tutte polynomial of $K_{2,m}$ could be easily computed by using the general formula for the Tutte polynomial of the tensor product of graphs G_1 and G_2, where, G_1 is the graph P_2^m , that is m parallel edges, and G_2 is the path P_3 of length 2. This particular case of the tensor product is also referred to as the stretching of G_1, see [10]. We get that

$$T(K_{2,m};x,y) = (x+1)^{m-1} \left(\sum_{i=1}^{m-1} \left(\frac{x+y}{x+1} \right)^i + x^2 \right). \qquad (3-3)$$

By evaluating the Tutte polynomial of G at $(1,1),(0,2)$ and $(2,0)$ we obtain $\tau(G) = 2^{m-1}(2+m)$, $\alpha^*(G) = 2^{m+1}-2$ and $\alpha(G) = 2\times 3^m$. So, we want to prove that $6^m - 1 \geq 2^{2m}(\frac{m+2}{4})^2$. It is enough to prove that $m\ \ln(3/2) > 2\ln(\frac{m+2}{4})$. But basic calculus shows that the function $\ln(3/2)x - 2\ln(\frac{x+2}{4})$ is positive in the interval $[1,1)$. $\square$

Let $n\geq 4$, $m\geq 0$ and let G be a threshold graph with degree sequence $(d_1,\cdots,d_{n+m})$ such that $d_1 = d_2 = n+m-1$, $d_3 = \cdots = d_n = n-1$, and $d_{n+1} = \cdots = d_{n+m} = 2$. So, the graph G is the parallel connection of K_n and the graph $K^+_{2,m}$ along e_0. We denote these graphs by $\Pi(n,m)$.

Lemma 3.4 *For $n\geq 4$ and $m\geq 0$, $\gamma(\Pi(n,m))\geq \tau^2(\Pi(n,m))$.*

Proof For $n=4$ and $m\geq 0$ we can use the formula for computing the Tutte polynomial of the parallel connection given in [2] to get an expression for the number of spanning trees, acyclic orientations and totally cyclic orientations. Thus, $\tau(\Pi(4,m)) = 2^{m+2}(4+m)$, $\alpha(\Pi(4,m)) = 8\times 3^{m+1}$, $\alpha^*(\Pi(4,m)) = 9\times 2^{m+2}-12$. Then, $\gamma(\Pi(4,m)) > \tau^2(\Pi(4,m))$ for $m\geq 0$. By formulae $(3-1)$ and $(3-2)$ and Lemma 3.2, we have the following recursive relations for acyclic and totally cyclic orientations: $\alpha(\Pi(n+1,m)) = (n+1)\alpha(\Pi(n,m))$, $\alpha^*(\Pi(n+1,m)) \geq (2^n-2)\alpha^*(\Pi(n,m))$ and for trees we have

$$\tau(\Pi(n+1,m)) = \left(\frac{n+m+1}{n+m}\right)\left(\frac{n+1}{n}\right)^{n-3}(n+1)\tau(\Pi(n,m)).$$

Using these relations we get the following inequalities:

$$\begin{aligned}
\ln(\tau^2(\Pi(n+1,m))) &= \ln(\tau^2(\Pi(n,m))) + 2\ln\left(\frac{n+m+1}{n+m}\right)\\
&\quad +2(n-3)\ln\left(\frac{n+1}{n}\right) + 2\ln(n+1)\\
&\leq \ln(\tau^2(\Pi(n,m))) + 2(n-2)\ln\left(\frac{n+1}{n}\right) + 2\ln(n+1)\\
&\leq \ln(\tau^2(\Pi(n,m))) + \ln(2^n-2) + 2\ln(n+1).
\end{aligned}$$

The last inequality can be proved using Lemma 3.1 for $n\geq 5$ and by direct computation for $n=4$. Now, by using induction we have that the last term is at most $\ln((\Pi(n,m))) + \ln(2^n - 2) + \ln(n+1)$. Algebraic manipulation and the recursive relations mentioned above give that the last quantity is at most $\ln((\Pi(n+1,m)))$. $\square$

For $m=0$, a similar proof as above gives

Theorem 3.5 *For $n\geq 4$, $\alpha^*(K_n) > \tau(K_n)$.*

The quantity $\alpha^*(K_n)$ has also the interpretation of being the number of strongly connected labeled tournaments on n nodes, and its exponential generating function is given by $1-1/(1+f(x))$ where

$$f(x) = \sum_{m \geq 1} 2^{\binom{n}{2}} x^m/m!,$$

see [11] for details.

Finally, if G is a threshold graph with degree sequence $(d_1, \cdots, d_n, 2, \cdots, 2)$ and we delete the vertex labeled $n \geq 3$ with degree $d_n = p$, we obtain the (unique up to isomorphisms) threshold graph $G' = G - \{n\}$ with degree sequence $(d_1 - 1, \cdots, d_p - 1, d_{p+1}, \cdots, d_{n-1}, 2, \cdots, 2)$. If G is not the complete graph, then $\omega(G) = \omega(G - \{n\})$.

Lemma 3.6 *If G is a threshold graph with degree sequence $(d_1, \cdots, d_n, 2, \cdots, 2)$ and $d_n \geq 3$, then $\gamma(G) \geq \tau^2(G)$.*

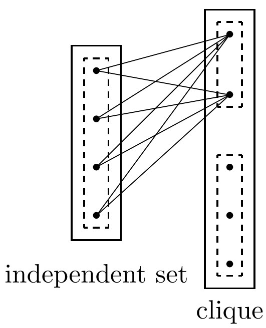

Fig.3 The case of Lemma 3.4.

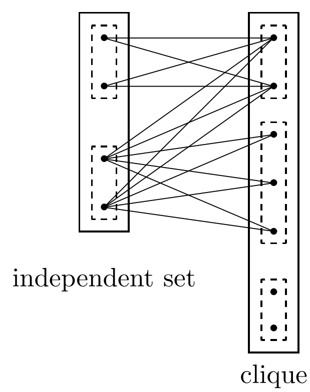

Fig.4 The case of Lemma 3.6.

Proof If $n = 2$, and thus $\omega(G) = 3$, then the result is true by Lemma 3.3. If $d_n = n - 1$, and thus $\omega(G) = n$, the result follows by Lemma 3.4. We assume that $d_n = p < n - 1$.

We have, by formula $(3-1)$, the recursive relation $\tau(G) = p(\frac{n}{n-1})^{p-1}\tau(G')$. Thus we obtained the following inequalities:

$$\begin{aligned}
\ln(\tau^2(G)) &= \ln(\tau^2(G')) + 2(p-1)\ln\left(\frac{n}{n-1}\right) + \ln(p^2) \\
&< \ln(\tau^2(G')) + \frac{6(p-1)}{3n-2} + \ln(p^2) \\
&< \ln(\tau^2(G')) + \frac{6(p-1)}{3p+4} + \ln(p^2) \\
&< \ln(\tau^2(G')) + \ln(2^p - 2) + \ln(p+1),
\end{aligned}$$

where the second inequality follows by Lemma 3.1, the third inequality is because $p \leq n-2$, and the last inequality is because the function $\ln(2^x - 2) + ln(p+1) - \frac{6(x-1)}{3x+4} - \ln(x^2)$ is positive in $[3, \infty)$ which can be proved by basic calculus. Using induction we have that the last expression is at most $\ln(\gamma(G')) + \ln(2^p - 2) + ln(p+1)$. But by formulae $(3-2)$ and Lemma 3.2 we have

$\alpha(G) = (p+1)\alpha(G')$ and $\alpha^*(G) \geq (2^p-2)\alpha^*(G')$. Therefore, $\ln(\gamma(G'))+\ln(2^p-2)+ln(p+1) \leq \ln\gamma(G)$. $\square$

Theorem 3.7 *If G is a threshold graph, then* $\max\{\alpha(G), \alpha^*(G)\} \geq \tau(G)$.

It is not difficult to modify the proof of the previous theorem to get the following

Theorem 3.8 *If G is a threshold graph with degree sequence $(d_1, \cdots, d_n)$ and $d_n \geq 3$, then $\alpha^*(G) \geq \tau(G)$.*

We believe that the same techniques can be used to prove that the conjecture is true for chordal graphs.

3.2 Complete bipartite graphs

The complete bipartite graphs $K_{n,m}$ also satisfy the conjecture. Using the matrix-tree theorem it is easy to prove that the number of spanning trees of $K_{n,m}$ is $n^{m-1}m^{n-1}$. For $K_{2,m}$ equation $(3-3)$ can be used to compute the number of acyclic orientations, and it can be checked that $\tau(K_{2,m}) = 2^{m-1}m \leq 2 \times 3^m - 2^m = \alpha(K_{2,m})$ for $m \geq 2$.

The exponential generating function for the chromatic polynomial of $K_{n,m}$ appears in [20]. Using the same technique the authors in [13] gave the exponential generation function for the Tutte polynomial of $K_{n,m}$. From that we get an expression for the exponential generating function for totally cyclic orientations,

$$1 - \sum_{(n,m)\neq(0,0)\in N^2} \alpha^*(K_{n,m})\frac{x^n}{n!}\frac{y^m}{m!} = \left(\sum_{(n,m)\in N^2} 2^{nm}\frac{x^n}{n!}\frac{y^m}{m!}\right)^{-1}.$$

By grouping terms, the right hand side can be written as the inverse of the exponential generating function $e^y + e^{2y}x + e^{4y}x^2/2! + \cdots$. By solving we get $e^{-y} - x + (2e^y - e^{2y})x^2/2! + (6e^{3y} - e^{6y} - 6e^{2y})x^3/3! + \cdots$. Thus, $\alpha^*(K_{1,m}) = 0$ for all $m \geq 1$; $\alpha^*(K_{2,m}) = 2^m - 2$ for $m \geq 1$ and $\alpha^*(K_{3,m}) = 6^m - 6 \times 3^m + 6 \times 2^m$ for $m \geq 1$. We conclude that $\alpha^*(K_{3,m})$ is bigger than $\tau(K_{3,m})$ for $m \geq 4$.

For $n \geq 3$ we have the following recursive relations:

$$\tau(K_{n+1,m}) = m(1+\frac{1}{n})^{m-1}\tau(K_{n,m})$$

and

$$\alpha^*(K_{n+1,m}) \geq (2^m-2)\alpha^*(K_{n,m}).$$

As $n \geq 3$, $\ln(1+1/n) \leq 3/(3n+1) \leq 3/10$ by Lemma 3.1. Thus, $\tau(n+1,m) \leq \alpha^*(n+1,m)$ follows by induction as $\ln m + 3/10(m-1) \leq \ln(2^m-2)$ for $m \geq 4$. Putting these together we have the following.

Theorem 3.9 *For all $m \geq n \geq 2$,* $\max\{\alpha(K_{n,m}), \alpha^*(K_{n,m})\} \geq \tau(K_{n,m})$.

§4. Experimental data

Our experiments are based on two computer programs. One is **TuLiC** written in Java to compute the Tutte polynomial for moderate-sized graphs. The program was developed by Conde using the algorithm given in [18]. The program can be found at

http://ada.fciencias.unam.mx/ rconde/tulic/.

The second one is the **Nauty** package of Brendan McKay which we use to generate all non-isomorphic 2-connected simple graphs, see [12]. We automatized the search for a counterexample to Conjecture 2.1 by the following steps:

1. We generated eight files, each one containing all the non-isomorphic 2-connected simple graphs of $3, 4, 5, 6, 7, 8, 9$ and 10 vertices respectively, in a simple format that is easily parsed.
2. We wrote a program in the Java language that uses **TuLiC**'s libraries to evaluate the Tutte polynomial of each graph at the points $(1, 1), (2, 0)$ and $(0, 2)$. The program enumerates all the graphs and the output for each graph is of the form *graph-number* $\tau(G)\alpha(G)\alpha^*(G)$ [*true*|*false*].
3. Finally, we made a shell script that uses the program described above to execute all the experiments. The script needs at least two parameters: The input file, containing the graphs to be tested, and the output file, where it puts the results.

For example, the output file at the end of the process for all the 2-connected simple graphs on 5 vertices will look like this:

1 :	12	46	6	true
2 :	20	54	14	true
3 :	5	30	2	true
4 :	11	42	6	true
5 :	21	54	18	true
6 :	40	72	60	true
7 :	24	60	24	true
8 :	45	78	78	true
9 :	75	96	204	true
10 :	125	120	544	true

To find a counterexample for the conjecture, we only needed to search these files for a line containing the word *false*. With the graph number, the graph can be extracted from the original file. After we ran the experiments on all the graphs, in total $9,945,269$, no counterexample was found.

§5. Conclusion

We proved Conjecture 2.1 for some infinite families of graphs and so we have provided evidence that makes the conjecture more plausible. There are some more families; for example Marc Noy

(private communication) proved that $\tau(G) \leq \alpha(G)$ when G is a maximal outerplanar graph by using contraction-deletion.

However, we were unable to prove anything relevant for graphs with multiple edges or, the dual concept, with edges in series. It remains to be discovered if by checking graphs with a good mixed of multiple edges and edges in series a counterexample can be found.

There are several related problems. For example, the complexity of deciding if $\alpha(G)$ is bigger than $\tau(G)$ is not clear. Computing exactly $\alpha(G)$ is #P-complete, see [10], but this is much more than is needed. In [4] an FPRAS for $\alpha(G)$ is given provided G has girth at least $(5+\delta)log_2 n$. So, for this class of graphs, there is an FPRAS for the quantity $\alpha(G) - \tau(G)$.

Our results about complete bipartite graphs suggest that Conjecture 2.1 may be true for graphs that either contain two edge-disjoint spanning trees or in which the edge-set is the union of two spanning trees. Probing the conjecture for 4-edge-connected graphs would be a good first step but we were not able to do that. However, this can be proved for a similar class of graphs. If G is an n-vertex m-edge graph with edge-connectivity $\Omega(log(n))$, then $\alpha^*(G) = 2^m(1-O(1/n))$ by a result in [9]. We also have a general upper bound for $\tau(G)$ given in [8]; that is,

$$\tau(G) \leq \left(\frac{n}{n-1}\right)^{n-1} \left(\frac{\prod_i d_i}{2m}\right)$$

where $\prod_i d_i$ is the product of the vertex-degrees. Since 2^m is bigger than the last quantity, this class of graphs also satisfies the conjecture.

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DECOMPOSITION OF GRAPHS INTO INTERNALLY DISJOINT TREES

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Abstract: A Smarandache graphoidal tree (k, d)-cover of a graph G is a partition of edges of G into trees $T_1, T_2, \cdots, T_l$ such that $|E(T_i) \cap E(T_j)| \leq k$ and $|T_i| \leq d$ for integers $1 \leq i, j \leq l$. In this paper we investigate the garphoidal tree covering number $\gamma_T(G)$, i.e., Smarandache graphoidal tree $(0, \infty)$-cover of complete graphs, complete bipartite graphs and products of paths and cycles. In [5] M.F.Foregger, define a parameter $z'(G)$ as the minimum number of subsets into which the vertex set of G can be partitioned so that each subset induces a tree. In this paper we also establish the relation $z'(G) \leq \gamma_T(G)$.

Key Words: Smarandache graphoidal tree (k, d)-cover, graphoidal tree cover, complete graph, complete bipartite graph, product of path and cycle.

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§1. Introduction

By a graph we mean a finite, undirected graphs without loops and multiple edges. Terms not defined here are used in the sense of Harary [6]. Any vertex of a graph H of degree greater than 1 is called an internal vertex of H. A *Smarandache graphoidal tree* (k, d)*-cover* of a graph G is a partition of edges of G into trees $T_1, T_2, \cdots, T_l$ such that $|E(T_i) \cap E(T_j)| \leq k$ and $|T_i| \leq d$ for integers $1 \leq i, j \leq l$. Particularly, a Smarandache graphoidal tree $(0, \infty)$-cover, usually called a *graphoidal tree cover* of G is a collection of non C trivial trees in G such that

(i) every vertex is an internal vertex of at most one tree;

(ii) every edge is in exactly one tree.

Let $\mathscr{G}$ denote the set of all graphoidal tree covers of G. Since $E(G)$ is a graphoidal tree cover, we have $\mathscr{G} \neq \emptyset$. We define the graphoidal tree covering number of a graph G to be the minimum number of trees in anygraphoidal tree cover of G, and denote it by $\gamma_T(G)$. Any

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graphoidal tree cover $\mathscr{J}$ of G for which $|\mathscr{J}| = \gamma_T(G)$ is called a minimum graphoidal tree cover.

Example 1.1 Consider a graph G given in the Fig.1.

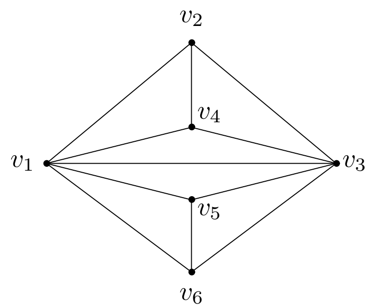


Fig.1

Let T_1, T_2 and T_3 be the trees in Fig.2.

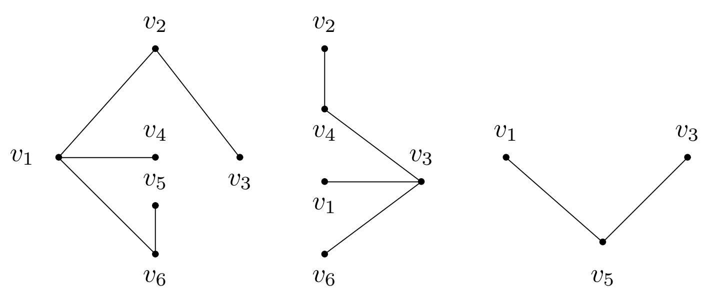


Fig.2

It is easy to see that the graph G in Fig.1 cannot be covered by two trees. Since the three trees shown in Fig.2 form a graphoidal tree cover, $\gamma_T(G) = 3$.

Observation 1.2 If $\deg(v) > \gamma_T(G)$, then v is an internal vertex in some tree in every minimum graphoidal tree cover.

Observation 1.3 For a (p, q) graph G, $\gamma_T(G) \geq \lceil \frac{q}{p-1} \rceil$.

Observation 1.4 $\gamma_T(G) > \frac{\delta(G)}{2}$ if $\delta(G) > 0$.

§2. Preliminaries

$\tau(G)$ is the minimum number of subsets into which the edge set $E(G)$ of G can be partitioned so that each subset forms a tree. A cyclically 4-edge connected graph is one in which the removal of no three edges will disconnect the graph into two components such that each component

contains a cycle. We state some preliminary results from [2] and [4].

Theorem 2.1([4]) $\tau(K_n) = \lceil \frac{n}{2} \rceil$.

Theorem 2.2([2]) *If G is a 2-connected cubic graph with p vertices, $p \geq 8$, then $\tau(G) \leq \lfloor \frac{p}{4} \rfloor$.*

Theorem 2.3([2]) *If G is a 3-connected cubic graph with p vertices, $p \geq 12$, then $\tau(G) \leq \lfloor \frac{p}{6} \rfloor$.*

Theorem 2.4([2]) *If G is a cyclically 4-edge connected cubic graph with p vertices, $8 \leq p \leq 16$ then $\tau(G) = 2$.*

§3. Complete and complete bipartite graphs

We first determine the graphoidal tree covering number of a complete graphs.

Theorem 3.1 $\gamma_T(K_n) = \lceil \frac{n}{2} \rceil$.

Proof From observation 1.4, it follows that $\gamma_T(K_n) \geq \lceil \frac{n}{2} \rceil$. We give a construction for the reverse inclusion. First, let n be even, say $n = 2k$. For $i = 1, 2, \cdots, k$, let T_i be the tree shown in Fig.3 (subscripts modulo n). The Standard Rotation Method shows that this is true. (see Fig.4 for the case $n = 8$).

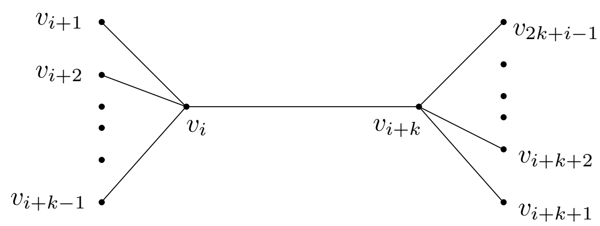


Fig.3

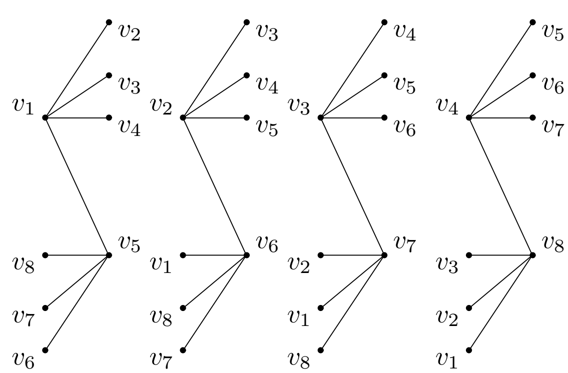


Fig.4

For n odd, delete one vertex from each tree in the decomposition given for K_{n+1}. The result is clearly a graphoidal tree cover for K_n, once isolated vertices are removed. □

We now turn to the case of complete bipartite graphs, beginning with a general result on the diameter of trees in a minimum graphoidal tree cover. The following standard notation is used for the partite sets of $K_{m,n}$ with $m \leq n$: $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$.

Lemma 3.2 *If a minimum graphoidal tree cover $\mathscr{J}$ of $K_{m,n}$ contains a tree with a path of length≥ 5, then it also contains a tree with exactly one edge.*

Proof Let $T \in \mathscr{J}$ contain a path $P = (x_1, y_1, x_2, y_2, x_3, y_3, \cdots)$ where $x_i \in X$ and $y_j \in Y$. Since y_1 and x_3 are internal in T, these cannot be internal in any other member of $\mathscr{J}$. Therefore $T_1 = \{(y_1x_3)\} \in \mathscr{J}$. □

Lemma 3.3 *If $m \leq n \leq 2mC\ 3$, $\gamma_T(K_{m,n}) \geq \lceil \frac{m+n}{3} \rceil$.*

Proof Suppose $\gamma_T(K_{m,n}) = r$ with $r < \lceil \frac{m+n}{3} \rceil$. Let $\mathscr{J}$ be a minimum graphoidal tree cover of $K_{m,n}$. Since $\delta(G) = m > \frac{m+n}{3} > r$ (as $n \leq 2mC\ 3$), by Observation 1.3, we have every vertex is an internal vertex of a tree in $\mathscr{J}$.

Claim 1. No tree in $\mathscr{J}$ can have more than two internal vertices from X with a common neighbor from Y. Suppose $x_1, x_2, \cdots, x_k$ $(k \geq 3)$ are all adjacent to y_1 in T_1 of $\mathscr{J}$. Then the sum of degrees of $x_1, x_2, \cdots, x_k$ in T_1 is at most $n + k - 1$. But each x_i $(i = 1, 2, \cdots, k)$ is an end vertex in at most $r-1$ other members of $\mathscr{J}$. So they have at most $n+k-1+k(r-1)$ total adjacencies in $\mathscr{J}$. Since $r < \frac{m+n}{3}$, $n+k-1+k(r-1) < \frac{3n+3k-3+k(m+n)-3k}{3} \leq \frac{n(2k+3)}{3} - 1$ $(m \leq n)$ $= nk - (\frac{n(k-3)}{3} + 1) < nk$ $(k \geq 3)$, a contradiction. Hence we have Claim 1.

Claim 2 There exists a minimum graphoidal tree cover $\mathscr{J}'$ such that no tree in $\mathscr{J}'$ has a path of length≥ 5.

Suppose $T_1 \in \mathscr{J}$ has a path $(x_1, y_1, x_2, y_2, x_3, y_3, \cdots)$. Then by the previous lemma, a tree T_2 in $\mathscr{J}$ has just the single edge y_1x_3. Let T_1' be the tree containing x_2 obtained by removing the edge y_1x_2 from T_1. Let T_2' be the tree $(T_1 - T_1') \cup T_2$. Let $\mathscr{J}_1'$ be the graphoidal tree cover obtained from $\mathscr{J}$ after replacing T_1, T_2 by T_1', T_2' respectively. If there is a tree in $\mathscr{J}_1'$ again contains a path of length≥ 5 we repeat this process for $\mathscr{J}_1'$ and so on. Finally we get the required minimum graphoidal tree cover $\mathscr{J}'$. Hence we get Claim 2.

Now we can assume that no tree in $\mathscr{J}$ has a path of length ≥ 5.

Claim 3 No tree in $\mathscr{J}$ can have more than two internal vertices from Y with a common neighbor from X.

Suppose there is a tree T_1 in $\mathscr{J}$ containing k internal vertices $y_1, y_2, \cdots, y_k$ $(k \geq 3)$ with a common neighbor x_1. Since $m > \gamma_T(G)$ and every vertex is an internal vertex of a tree in $\mathscr{J}$, there is a tree in $\mathscr{J}$, say, T_2 containing at least two vertices from X as internal vertices. By Claim 2, the internal vertices of a tree in $\mathscr{J}$ form a star and so the internal vertices from X in T_2 have a common neighbor from Y. By Claim 1, T_2 has exactly two internal vertices x_2 and x_3 from X with a common neighbor y_s from Y. Between x_2, x_3 and $y_1, y_2, \cdots, y_k$ there are $2k$ edges in $K_{m,n}$. Clearly x_2 and x_3 can be made adjacent with two y's in T_1. Let

it be y_1 and y_2. Now $y_1, y_2, \cdots, y_k$ can be made adjacent with x's in T_2. But it will cover exactly $k+2$ edges (out of $2k$ edges) and so by the definition of graphoidal tree cover, each uncovered edge is a tree in $\mathscr{J}$. Without loss of generality let $T_3, T_4, \cdots, T_k$ be the trees with edges $(y_3, x_{l_3}), (y_4, x_{l_4}), \cdots, (y_k, x_{l_k})$ respectively, where $l_i \in \{2,3\}$, $3 \le i \le k$. By Claim 2 the internal vertices of T_1 form a star. Removing all the edges incident with y_i from T_1 to form the tree T_i' $(3 \le i \le k)$. Let T_1' be the tree formed by the remaining edges of T_1 after the removal. Now each T_i in $\mathscr{J}$ is replaced by $T_i' \cup T_i$ for $3 \le i \le k$. Also replace T_1 in $\mathscr{J}$ by T_1'. If $\mathscr{J}$ again contains a tree having more than two internal vertices from Y with a common neighbor from X. We repeat the above process and so on. Hence we have Claim 3. From Claims 1, 2 and 3, it follows that no tree in $\mathscr{J}$ has more than three internal vertices. Since every vertex of $K_{m,n}$ must be an internal vertex of a tree in $\mathscr{J}$ and $\gamma_T(K_{m,n}) = r$, we have only $3r$ ($< m+n$) internal vertices in $\mathscr{J}$. This is a contradiction. Hence $\gamma_T(K_{m,n}) \ge \lceil \frac{m+n}{3} \rceil$. □

Theorem 3.4 *If $m \le n \le 2mC\ 3$, then $\gamma_T(K_{m,n}) = \lceil \frac{m+n}{3} \rceil$. Furthermore, if $n > 2m-3$, then $\gamma_T(K_{m,n}) = m$.*

Proof By Lemma 3.3, $\gamma_T(K_{m,n}) \ge \lceil \frac{m+n}{3} \rceil$. Next we proceed to prove $\gamma_T(K_{m,n}) \le \lceil \frac{m+n}{3} \rceil$, where $3 \le m \le n \le 2m-3$. Let $r = \lfloor \frac{2mn}{3} \rfloor = \frac{2m-n+k}{3}$ where k is $0, 1$ or 2.

Define for $1 \le i \le r$

$P_i = \{(x_i, y_i x_{r+i})\} \cup \{(y_i, x_j) : j \ne i, r+i; 1 \le j \le m-k\} \cup \{(x_i, y_j) : r < j \le m-r-k\} \cup \{(x_{r+i}, y_j) : m-r-k < j \le n-k\}$.

For $1 \le i \le m-2r-k$ we define

$P_{i+r} = \{(y_{r+i}, x_{2r+i}, y_{m-r-k+i})\} \cup \{(x_{2r+i}, y_j); j \ne r+i, m-r-k+i, r < j \le n-k\} \cup \{(y_{r+i}, x_j) : r+1 \le j \le 2r\} \cup \{(y_{m-r-k+i}, x_j) : 1 \le j \le r\}$.

For $k = 1$

$P_{m-r} = \{(x_m, y_n)\} \cup \{(x_m, y_j) : 1 \le j \le n-1\} \cup \{(y_n, x_j) : 1 \le j \le m-1\}$.

For $k = 2$

$P_{m-r-1} = \{(x_{m-1}, y_{n-1})\} \cup \{(x_{m-1}, y_j) : 1 \le j \le nC\ 2\} \cup \{(y_{n-1}, x_j) : 1 \le j \le mC\ 2\}$,

$P_{m-r} = \{(x_m, y_n)\} \cup \{(x_m, y_j) : 1 \le j \le n-1\} \cup \{(y_n, x_j) : 1 \le j \le m-1\}$.

Clearly $\mathscr{J} = \{P_1, P_2, \cdots, P_{m-r}\}$ is a graphoidal tree cover for $K_{m,n}$. Therefore $\gamma_T(K_{m,n}) \le m - r = \frac{m+n+k}{3} \le \lceil \frac{m+n}{3} \rceil$.

Let $n = 2m-2+k$, $k \ge 0$. Suppose that $\gamma_T(K_{m,n}) \ne m$. Then there exists a graphoidal tree cover $\mathscr{J}$ with at most $m-1$ trees. Since $\delta(G) > m-1$, it follows that every vertex is an internal vertex of a tree in $\mathscr{J}$. If x_i is an internal vertex of a tree T in $\mathscr{J}$ then $\deg_T(x_i) \ge 2m-2+k-(m-2) = m+k$. This implies that in any minimum graphoidal tree cover exactly one vertex of X should be internal in a tree. But there are m vertices and $|\mathscr{J}| \le m-1$. This leads to a contradiction. Hence $\gamma_T(K_{m,n}) \ge m$. Clearly, $\gamma_T(K_{m,n}) \le m$ and so

$$\gamma_T(K_{m,n}) = m.$$ □

The following examples illustrate $\gamma_T(K_{m,n}) = \lceil \frac{m+n}{3} \rceil$.

Example 3.5 (i) Consider $K_{4,5}$. Clearly $k = 0$ and $r = 1$.

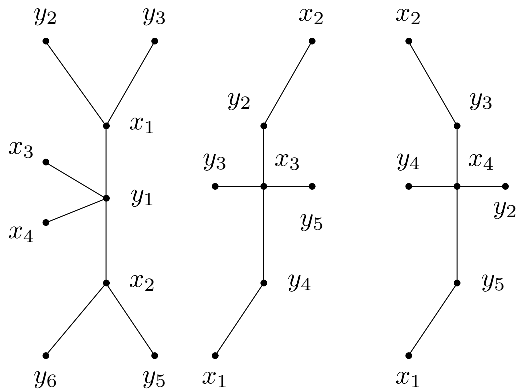


Fig.5

(ii) Consider $K_{8,9}$. Clearly $r = 2$ and $k = 1$.

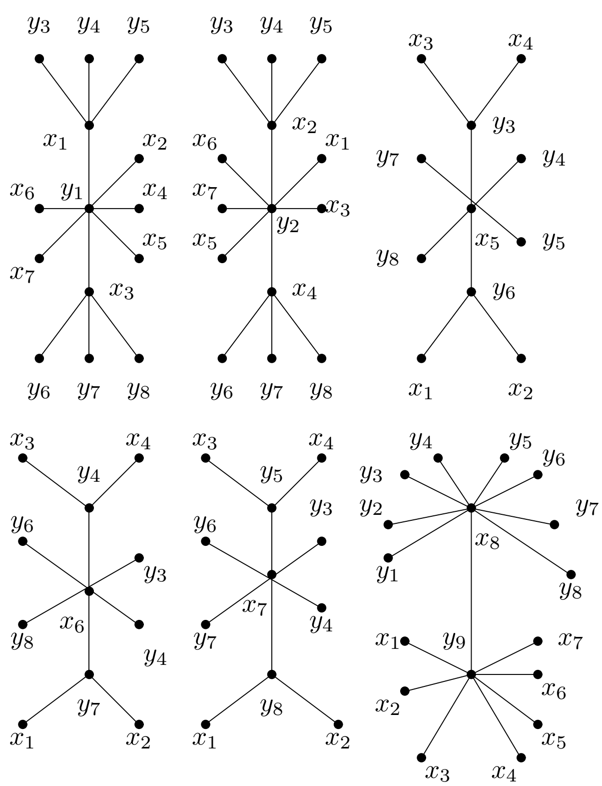


Fig.6

(iii) Consider $K_{12,13}$. Clearly $r = 3$ and $k = 2$.

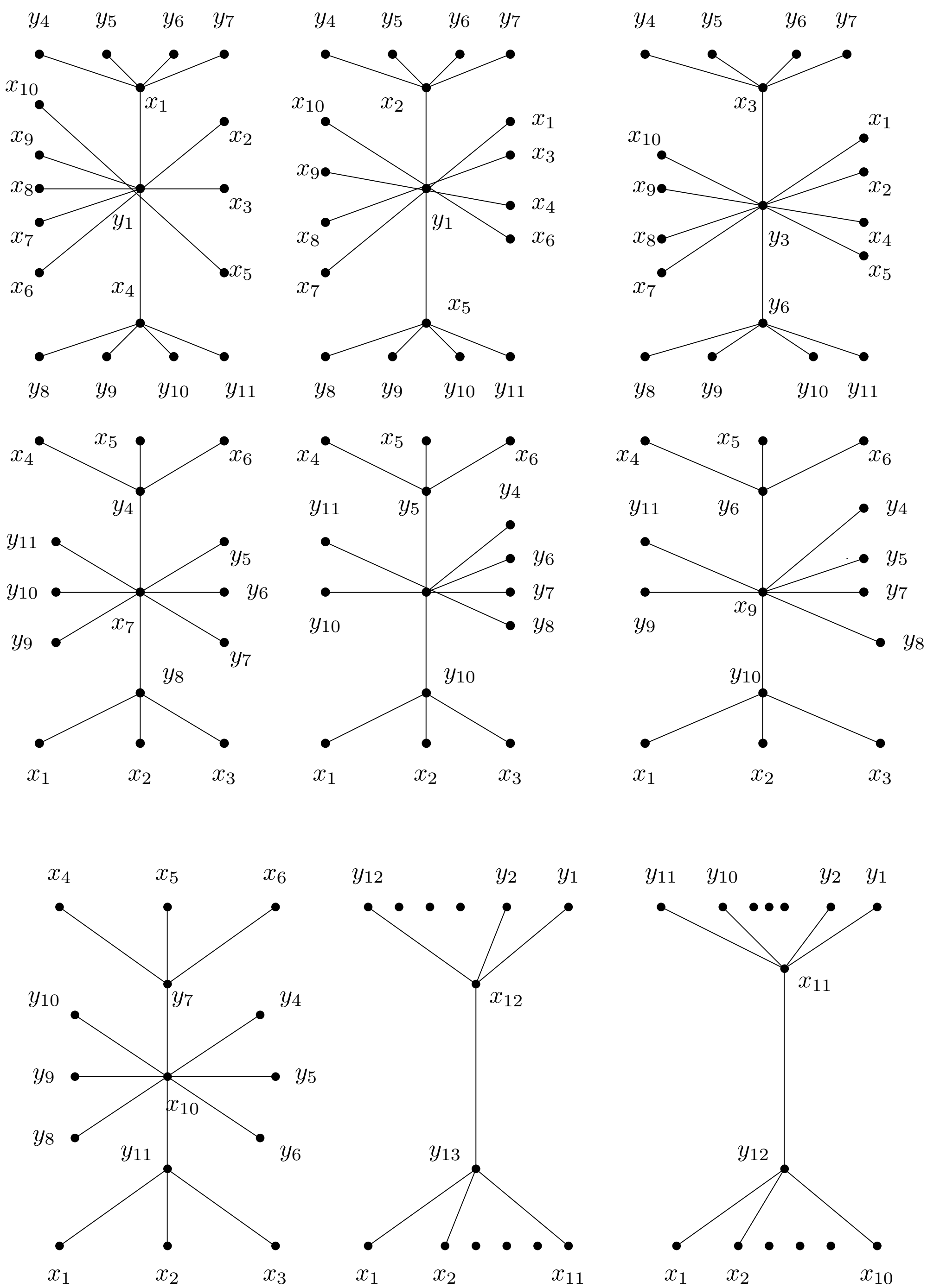


Fig.7

§4. Product of paths and cycles

Theorem 4.1 *(a)* $\gamma_T(P_m \times P_n) = 2$ *for integers* $m, n \geq 2$*;*

(b) $\gamma_T(P_n \times C_m) = 2$ *for integers* $m \geq 3$*,* $n \geq 2$*;*

(c) $\gamma_T(C_m \times C_n) = 3$ *for integers* $m, n \geq 3$*.*

Proof The fact that $P_m \times C_{2r}$ and $P_m \times C_{2r+1}$ can be decomposed into graphoidal tree covers of order 2 clearly follows from Fig.11 and Fig.12. Hence (b) follows. It is easily seen that deleting the edges (W_{i1}, W_{in}), $i = 1, 2, 3, \cdots, m$ (from Fig.8, Fig.9 and Fig.10) produces a graphoidal tree cover for $P_m \times P_n$ and so (a) follows. Since $C_m \times C_n$ is 4-regular, $\gamma_T(C_m \times C_n) > 2$. Now consider the minimum graphoidal tree cover $\{T_1, T_2\}$ of $P_{m-1} \times C_n$.

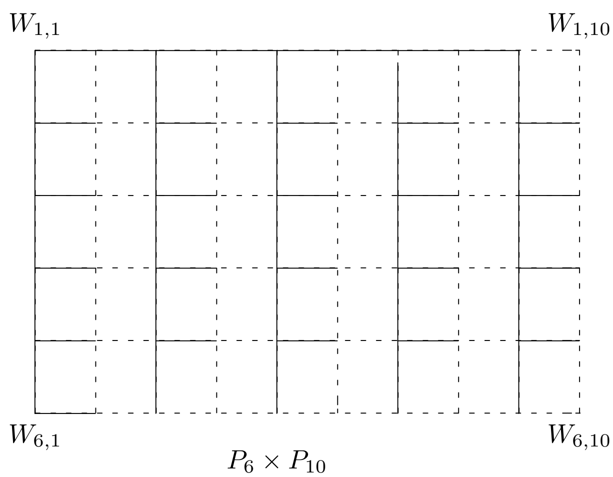


Fig.8

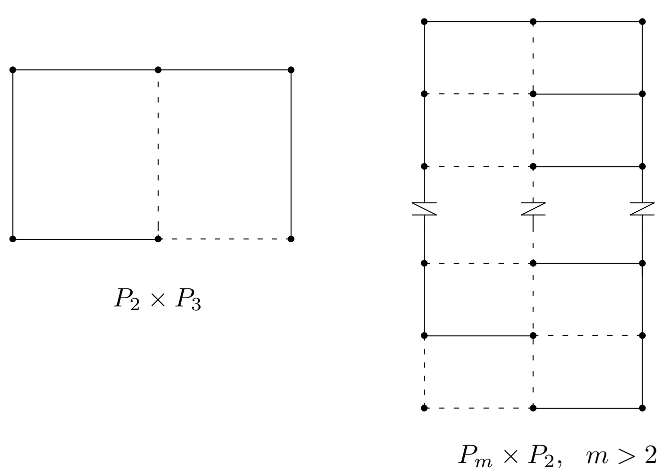
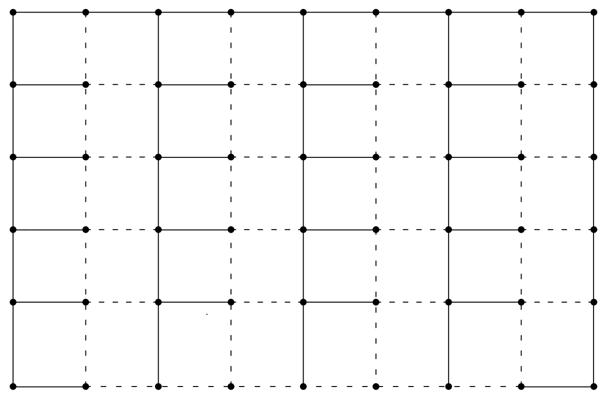


Fig.9



$P_6 \times P_9$

Fig.10

Here in Fig.11 and Fig.12 thick lines form the tree T_1 and dotted lines form the tree T_2.

Case (i) *n is even.*

To the tree T_2, add vertices $w_{m,2}$ and $w_{m,4}$. Then add a vertex $w_{m,3}$ adjacent to $w_{1,3}$, $w_{m-1,3}$, $w_{m,2}$ and $w_{m,4}$. The only additional internal vertex this creates is $w_{m,3}$.

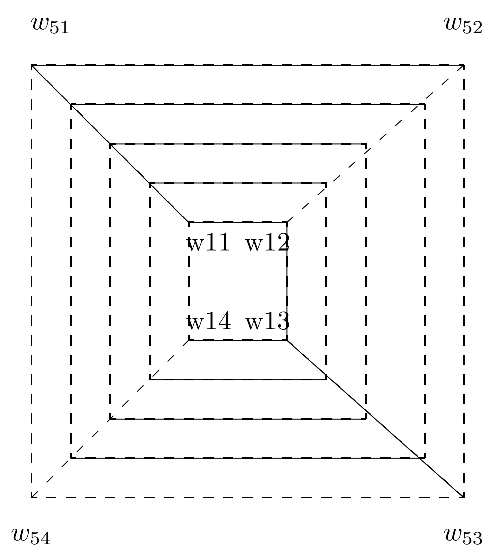
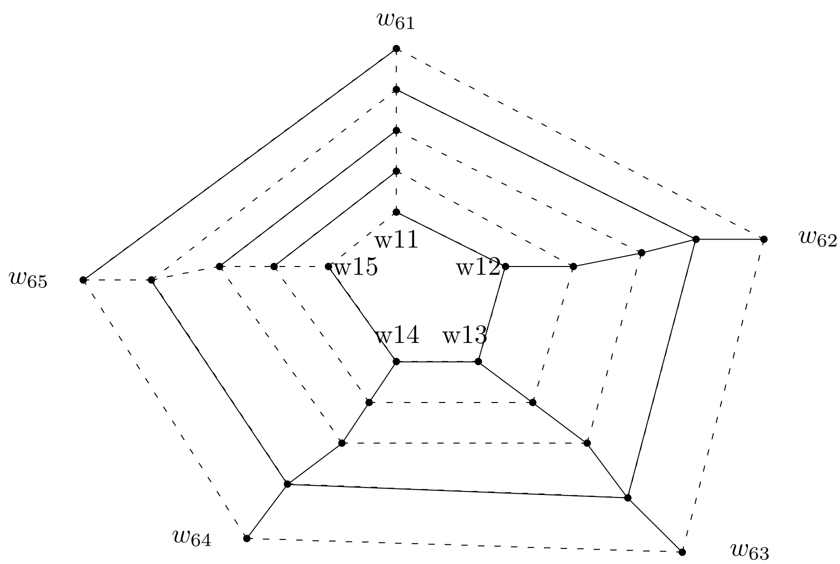


Fig.11

Now take a third tree as $T_3 = \{(w_{m,i-1}, w_{m,i}) : 2 \leq i \leq n; i \neq 3, 4\} \cup \{(w_{m,1}, w_{m,n})\} \cup \{(w_{m,i}, w_{1,i}), (w_{m,i}, w_{m-1,i}) : 1 \leq i \leq n; i \neq 3\}$.

Case (ii) *n is odd.*

To the tree T_2, add vertices $w_{m,1}$ and $w_{m,3}$. Then add a vertex $w_{m,2}$ adjacent to $w_{1,2}$, $w_{m-1,2}$, $w_{m,1}$, $w_{m,3}$. The only additional internal vertex this creates is $w_{m,2}$.



$P_6 \times C_5$

Fig.12

Now take a third tree as $T_3 = \{(w_{m,i-1}, w_{m,i}) : 4 \leq i \leq n\} \cup \{(w_{m,1}, w_{m,n})\} \cup \{(w_{m,i}, w_{1,i}), (w_{m,i}, w_{m-1,i}) : 1 \leq i \leq n; i \neq 2\}$. Now $\{T_1, T_2, T_3\}$ is a graphoidal tree cover for $C_m \times C_n$ and so $\gamma_T(C_m \times C_n) = 3$.

Observation 4.2 It is observed that $\tau(G) \leq \gamma_T(G)$. From Theorems 2.1 and 3.1, it follows that (i) $\tau(K_n) = \gamma_T(K_n) = \lceil \frac{n}{2} \rceil$. It is also observed that $\tau(G) = \gamma_T(G)$ for all graphs with maximum degree≤ 3. From Theorems 2.2, 2.3 and 2.4, it follows that

Theorem 4.3 *If G is a 2-connected cubic graph with p vertices, $p \geq 8$, then $\gamma_T(G) \leq \lfloor \frac{p}{4} \rfloor$.*

Theorem 4.4 *If G is a 3-connected cubic graph with p vertices, $p \geq 12$, then $\gamma_T(G) \leq \lfloor \frac{p}{6} \rfloor$.*

Theorem 4.5 *If G is a cyclically 4-edge connected cubic graph with p vertices, $8 \leq p \leq 16$ then $\gamma_T(G) = 2$.*

§5. Relationship between $\tau'(G)$ and $\gamma_T(G)$

In [5] Foregger, M F and Foregger, T.H defined $\tau'(G)$ as the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a tree. In this section we try to find some relationship between $\tau'(G)$ and $\gamma_T(G)$.

Theorem 5.1 *Let G be a graph with vertices $p \geq 4$ and let $\mathscr{J} = \{T_1, T_2, \cdots, T_n\}$ be a minimum graphoidal tree cover of G with $|E(T_j)| = 1$ for some j and $|E(T_i)| > 1$ for all $i \neq j$. Then we can always find a minimum graphoidal tree cover $\mathscr{J}' = \{T_1', T_2', \cdots, T_n'\}$ with $|E(T_i')| > 1$ for all i.*

Proof Let $T_j = \{(xy)\}$. Consider cases following.

Case (i) Suppose at least one of the vertices x and y, say x, is internal in a tree of $\mathscr{J}$. First assume that x is internal in a tree T_i of $\mathscr{J}$. If $y \notin V(T_i)$ then replacing T_i by $T_i \cup T_j$ and removing T_j from $\mathscr{J}$, we get a graphoidal tree cover $\mathscr{J}'$ with $|\mathscr{J}'| < |\mathscr{J}|$. Hence $y \in V(T_i)$. Let $(w, x, z, \cdots, y)$ be a path in T_i. Let C_1 and C_2 be the two components of $T_i - (xz)$ containing x and y respectively. Replace T_i and T_j by $C_2 \cup (xz)$ and $C_1 \cup T_j$ respectively so that both of them have at least two edges. Now $\mathscr{J}$ is still a minimum graphoidal tree cover and $|E(T)| > 1$ for every $T \in \mathscr{J}$.

Case (ii) Suppose both x and y are external vertices in $\mathscr{J}$. If $x \in V(T_i)$ and $y \notin V(T_i)$ then as in Case (i), we get a graphoidal tree cover $\mathscr{J}'$ with $|\mathscr{J}'| < |\mathscr{J}|$. Hence either $x, y \in V(T)$ or $x, y \notin V(T)$ for every T in $\mathscr{J}$. Let $x, y \in V(T_r)$, $T_r \in \mathscr{J}$. Suppose $|E(T_r)| > 2$. Let $e = (xz)$ be an edge in T_r. Replace T_r and T_j by $T_r - e$ and $T_j \cup e$ respectively and the result is true in this case. So let us assume that $|E(T_i)| = 2$ for some $T_i \in \mathscr{J}$ and $x, y \in V(T_i)$. Suppose $T_i = (xzy) \in \mathscr{J}$. Then $\deg(z) \geq 3$ in G. For, suppose $\deg(z) = 2$ in G. Since G is connected and $p \geq 4$, we must have at least one of the vertices x, y is of degree≥ 3. Since x or y alone can not be a member of a tree in $\mathscr{J}$ and $x, y \in V(T_i), V(T_j)$ we have $\deg(x) \geq 3$ and $\deg(y) \geq 3$.

Let x and y be external vertices in a tree T_r of $\mathscr{J}$ $(r \neq i, j)$. Replace T_r and T_j by $T_r \cup (xz)$ and $T_j \cup (zy)$ respectively. Now $\{T_1, T_2, \cdots, T_{i-1}, T_{i+1}, \cdots, T_n\}$ is clearly a graphoidal tree cover for G. This is a contradiction to the minimality of $\mathscr{J}$. Hence $\deg(z) \geq 3$ in G. Now, z must be external in some tree T_r of $\mathscr{J}$. Clearly $x, y \in V(T_r)$. Suppose $x, y \notin V(T_r)$. Replace T_r and T_j by $T_r \cup \{(xz)\}$ and $T_j \cup \{(zy)\}$ respectively in $\mathscr{J}$. Now $\{T_1, T_2, \cdots, T_{i-1}, T_{i+1}, \cdots, T_n\}$ is clearly a graphoidal tree cover for G. This is a contradiction to the minimality of $\mathscr{J}$. It shows that $x, y \in V(T_r)$. Since x, y and z are external vertices in T_r we have $|E(T_r)| \geq 3$. Let e be an edge in T_r containing z. Replace T_i and T_j by $\{T_i - (xz)\} \cup \{e\}$ and $T_j \cup \{(xz)\}$ respectively. Now $\mathscr{J}$ is a minimum graphodial tree cover and $|E(T)| > 1$ for every $T \in \mathscr{J}$. □

Proposition 5.2 *If $p \geq 4$, then there exists a minimum graphoidal tree cover of a connected graph G, in which every tree has more than one edge.*

Proof Let $\mathscr{J}$ be a minimum graphoidal tree cover of G and let $\mathscr{J} = \{T_1, T_2, \cdots, T_n\}$. Let us assume that $T_i = \{e_i\}, 1 \leq i \leq k$ and $|E(T_j)| > 1$ for $k+1 \leq j \leq n$. Let $G' = G - \{e_1, e_2, \cdots, e_k\}$. Clearly $\mathscr{J}' = \mathscr{J} - \{T_1, T_2, \cdots, T_k\}$ is a graphoidal tree cover for G'. Suppose G' is a disconnected graph. Then the number of components $\omega(G')$ is greater than one. If $\omega(G' \cup e_i) = \omega(G')$ for every $i \in \{1, 2, \cdots, k\}$ then G is disconnected. Hence we can choose $e_i = (x_i, y_i)$ for some $i \in \{1, 2, \cdots, k\}$ such that $\omega(G' \cup e_i) < \omega(G')$. Let G_1', G_1'' be the components of G' such that $G_1' \cup G_1'' \cup e_i$ is connected. Without loss of generality assume that $x_i \in G_1'$, $y_i \in G_1''$. If at all x_i is internal in a tree of $\mathscr{J}$, let it be in a tree T (of $\mathscr{J}$) in G_1'. Clearly $\mathscr{J}_1 = (\mathscr{J} - \{T, T_i\}) \cup \{T \cup T_i\}$ is a graphoidal tree cover of G

and $\mathscr{J}_1 < |\mathscr{J}|$. This is a contradiction. Hence G' is connected. Take $G_1 = G' \cup \{e_1\}$. Clearly $\mathscr{J}_1 = \mathscr{J}' \cup \{T_1\}$ is a minimum graphoidal tree cover for G_1 and $|\mathscr{J}_1| = n - k + 1$. For, suppose $\gamma_T(G_1) < n - k + 1$ and let $\mathscr{J}''$ be a minimum graphoidal tree cover for G_1. Then $|\mathscr{J}''| < n - k + 1$. Since $G = G_1 \cup \{e_2, \cdots, e_k\}$, $\mathscr{J}''' = \mathscr{J}'' \cup \{T_2, \cdots, T_k\}$ is a graphoidal tree cover for G and $|\mathscr{J}'''| = |\mathscr{J}''| + k - 1 < n - k + 1 + k - 1 = n$. This is a contradiction to the minimality of $\mathscr{J}$. Hence $\gamma_T(G_1) = n - k + 1$. By Theorem 5.1, there exists a minimum graphoidal tree cover $\mathscr{J}_1'$ of G_1 in which every tree has more than one edge and $|\mathscr{J}_1'| = |\mathscr{J}_1| = n - k + 1$. Let $G_2 = G_1 \cup \{e_2\}$. Proceeding as above, we find a minimum graphoidal tree cover $\mathscr{J}_2$ of G_2 in which every tree has more than one edge. Finally, we get $G = G_n = G_{n-1} \cup \{e_n\}$ and by a similar argument as above, we find a minimum graphoidal tree cover $\mathscr{J}_n$ of $G = G_n$ in which $|E(T)| > 1$ for every $T \in \mathscr{J}_n$. $\square$

Lemma 5.3 *Let $p(G) \geq 4$. Let $\mathscr{J}$ be a graphoidal tree cover of G such that $|E(T)| > 1$ for every tree $T \in \mathscr{J}$. Let $i(T)$ be the set of internal vertices of T. Then $\langle i(T) \rangle$-the subgraph induced by $i(T)$ is a subgraph of T and it is a tree for every $T \in \mathscr{J}$.*

Proof If $|i(T)| = 1$ then clearly the result is true. Let $|i(T)| > 1$. Let $x, y \in i(T)$ and $xy \in E(G)$. Suppose $xy \notin E(T)$. Then there exists T' of $\mathscr{J}$ such that $T' = \{(xy)\}$ by the definition of graphoidal tree cover. By our assumption this is not possible. Hence $\langle i(T) \rangle$ is a subgraph of T and it is a tree. Moreover, it is got by removing all the pendant vertices of T. $\square$

Theorem 5.4 *If G is a (p, q) graph with $p \geq 4$, then $\tau'(G) \leq \gamma_T(G)$.*

Proof By Proposition 5.2, we have known that result (1) following:

there exists a minimum graphoidal tree cover $\mathscr{J}$ such that $|E(T)| > 1$ for all $T \in \mathscr{J}$ and $|\mathscr{J}| = n$.

Let $\mathscr{J} = \{T_1, T_2, \cdots, T_n\}$.

Case (i) If every vertex is an internal vertex of a tree of $\mathscr{J}$, then $V(G) = i(T_1) \cup \cdots \cup i(T_n)$ is clearly a vertex partition of G. By Lemma 5.3, $\langle i(T_j) \rangle$ is a subgraph of T_j and is a tree for $1 \leq j \leq n$. Hence $\tau'(G) \leq n \leq \gamma_T(G)$.

Case (ii) Let x be one of the vertices which is not internal in any tree of $\mathscr{J}$. Let $x \in V(T_k)$ and $v \in i(T_k)$ such that $xv \in E(T_k)$. Since x is not internal in any tree of $\mathscr{J}$ and v is not internal in any tree except T_k, we have $\langle i(T_k) \cup \{x\} \rangle$ is a tree. For, if $xu \in E(G)$ and $xu \notin E(T_k)$ where $u \neq v$ in $i(T_k)$, then by the definition of graphoidal tree cover there exists T' of $\mathscr{J}$ such that $T' = \{(xu)\}$. This is a contradiction to claim in (1).

Let x, y be non-internal vertices in any tree of $\mathscr{J}$. Let $x, y \in V(T_k)$. If $xy \in E(G)$ then there exists T' of $\mathscr{J}$ such that $T' = \{(xy)\}$. This is a contradiction to the claim (1) also. Clearly, in this case $\langle i(T_k) \cup \{x, y\} \rangle$ is a tree. In this way we adjoin every such vertex to an $i(T_k)$. We make sure that each such vertex is adjoined to only one $i(T_k)$. These induced subgraphs give rise to a partition of $V(G)$ and these induced subgraphs form $n = \gamma_T(G)$ trees. Hence $\tau'(G) \leq n = \gamma_T(G)$. From Theorems 3.1 and 4.1 it follows that $\gamma_T(G) = \tau'(G)$ for the following graphs $K_n, P_m \times P_n$ and $P_n \times C_m$. $\square$

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Something attempted, something done.

By Menander, an ancient Greek dramatist.

Combinatorial Geometry with Application to Field Theory

(ISBN: 978-1-59973-100-1)

Written by Dr. Linfan Mao

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This book provides a survey of mathematics and physics by CC conjecture, i.e., *a mathematical science can be reconstructed from or made by combinatorialization*, formally presented by the author in 2006.

There are 8 chapters in this book. Chapters 1 and 2 are the fundamental of this book. Chapter 1 is a brief introduction to combinatorics with graphs and Chapter 2 is an application of combinatorial notion to mathematical systems. Algebraic structures, such as those of groups, rings, modules are generalized to a combinatorial one, particularly, a few well-known results in classical permutation groups are generalized to actions of multi-groups on finite sets.

Chapter 3 is a survey of Smarandache geometries. The topological spaces with fundamental groups, covering space and simplicial homology group, Euclidean spaces, differential forms in $\mathbf{R}^n$ and the Stokes theorem on simplicial complexes can be found in the first two sections. These Smarandache geometries, map geometries pseudo-Euclidean spaces, Smarandache manifolds, principal fiber bundles and geometrical inclusions in differential Smarandache geometries are established in the following.

Chapter 4 discusses topological behaviors of combinatorial manifolds with characteristics, such as Euclidean spaces and their combinatorial characteristics, vertex-edge labeled graphs, Euler-Poincaré characteristic, fundamental or singular homology groups on combinatorial manifolds and regular covering of combinatorial manifold by voltage assignment.

Chapters 5 and 6 form the main parts of combinatorial differential geometry. The former discusses tangent and cotangent vector space, tensor fields and exterior differentiation on combinatorial manifolds, connections and curvatures on tensors or combinatorial Riemannian manifolds, integrations and the generalization of Stokes' and Gauss' theorem, and so on. The later contains three parts. The first concentrates on combinatorial submanifold of smooth combinatorial manifolds with fundamental equations. The second generalizes topological groups to multiple one, for example Lie multi-groups. The third generalizes principal fiber bundles to combinatorial one by voltage assignment technique, which provides a mathematical fundamental for discussing combinatorial gauge fields.

Chapters 7 and 8 introduce the applications of combinatorial manifolds to fields. For this objective, variational principle, Lagrange equations in mechanical fields, Einstein's general relativity, Maxwell field and Yang-Mills gauge fields are introduced in Chapter 7. Chapter 8 generalizes fields to combinatorial fields under the *projective principle*, i.e., *a physics law in a combinatorial field is invariant under a projection on its a field*. Then, it show how to determine equations of combinatorial fields by Lagrange density, to solve equations of combinatorial gravitational fields and how to construct combinatorial gauge basis and fields, $\cdots$.

All material discussed in this book are valuable for researchers or postgraduates in combinatorics, topology, differential geometry, gravitational or quantum fields.

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